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Age-Dependent Branching Stochastic Processes in Cascade Theory - II (*).

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Summary. — In this paper we consider a simple model of an electron-photon cascade in which the transformation probabilities are functions of absorber depth. The mean and variance of the number of electrons in the cascade as a function of absorber depth are obtained and their properties discussed. An equation which yields the probability that the cascade will eventually terminate is also given.

1. – Introduction.

In a previous paper (1) we have presented an introduction to the Bellman-Harris theory of age-dependent stochastic processes (2), and considered a simple model of a cascade process based on this theory. L. JÁNOSSY (3) has employed an approach similar to that of Bellman and Harris in the derivation of the G-equations of cascade theory; and the same technique has been used by J. M. BLATT (4) in developing a stochastic theory of tracks in nuclear emul-

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(1) A. T. BHARUCHA-REID: *Phys. Rev.*, **96**, 751 (1954).

(2) R. BELLMAN and T. E. HARRIS: *Ann. Math.*, **55**, 280 (1958).

(3) L. JÁNOSSY: *Proc. Phys. Soc.* (*London*), A **63**, 241 (1950).

(4) J. M. BLATT: *Austral. Journ. Phys.*, **8**, 248 (1955).

sions. For a discussion of the Bellman-Harris theory, and related theories, we refer to reference ([5]).

In this paper we wish to consider a model for an electron-photon cascade in which the transformation probabilities are functions of absorber depth. In order to formulate this model it is necessary to consider a generalization of the Bellman-Harris theory, since these authors considered only constant transformation probabilities. This generalization has been obtained by W. A. O'N. WAUGH ([6]). The Waugh process can be described (in the nomenclature of cascade theory) as follows: Let $X(t)$ denote the number of electrons at absorber depth t in a one-dimensional electron-photon cascade, and let $G(\tau)$ denote the probability distribution of the absorber depth, say τ, travelled by an electron from its formation until it is transformed. Hence $G(\tau)$, $0 < \tau < \infty$, is the integral distribution for all paths of lengths less than or equal to τ. At the end of its path of travel the electron is transformed into n electrons ($n = 0, 1, 2, \ldots$) with probabilities $q_n(\tau)$, with each new electron having the same distribution $G(\tau)$ for the depth of the absorber it will travel before being transformed. For example: $q_0(\tau)$ is the probability of absorption; $q_1(\tau)$ is the probability that the electron is absorbed and one new electron formed, or that the electron continues to penetrate the absorber; and $q_2(\tau)$ is the probability that two new electrons will be formed, the original (parent) electron being absorbed, etc. In each case $q_n(\tau)$ is a prescribed function of τ; and these probabilities satisfy the condition $\sum_{n=0}^{\infty} q_n(\tau) = 1$ for all τ, $0 < \tau < \infty$.

Let $X(t)$ be an integer-valued random variable representing the number of electrons at depth t; hence the stochastic process $\{X(t),\ t \geqslant 0\}$ represents the development of the cascade. Now let $p(x, t) = \Pr\,(X(t) = x)$, $x = 0, 1, 2, \ldots$; and let

$$\pi(s, t) = \sum_{x=0}^{\infty} p(x, t) s^x , \qquad |s| \leqslant 1, \tag{1}$$

denote the generating function of the probabilities $p(x, t)$, starting with one incident electron at absorber depth zero. When the transformation probabilities depend on absorber depth, the generating function $\pi\,(s, t)$ satisfies the functional equation

$$\pi(s, t) = \int_0^t h[\pi(s, t - \tau), \tau]\, \mathrm{d}G(\tau) + s[1 - G(t)], \tag{2}$$

([5]) A. T. BHARUCHA-REID: *Elements of the Theory of Markov Processes and their Applications*, (New York, forthcoming).

([6]) W. A. O'N. WAUGH: *Biometrika*, **42**, 291 (1955).

where

$$h(s,\tau)=\sum_{n=0}^{\infty} q_n(\tau)s^n\,, \qquad |s|\leqslant 1 \tag{3}$$

that is, $h(s,\tau)$ is the generating function of the transformation probabilities $q_n(\tau)$.

Differentiation of (2) with respect to s yields integral equations of the renewal type for the moments of $X(t)$. For example, the mean or expected value of $X(t)$ is given by the solution of the integral equation

$$\mathcal{E}X(t)=m_1(t)=\int_0^t k_1(\tau)m_1(t-\tau)\,\mathrm{d}G(\tau)+1-G(t)\,, \tag{4}$$

where

$$k_1(\tau)=\left.\frac{\partial h(s,\tau)}{\partial s}\right]_{s=1}; \tag{5}$$

and the second derivative of $\pi(s,t)$, which we denote by $m_2(t)$, satisfies the integral equation

$$m_2(t)=\int_0^t k_2(\tau)[m_1(t-\tau)]^2\,\mathrm{d}G(\tau)+\int_0^t k_1(\tau)m_2(t-\tau)\,\mathrm{d}G(\tau)\,, \tag{6}$$

where

$$k_2(\tau)=\left.\frac{\partial^2 h(s,\tau)}{\partial s^2}\right]_{s=1}. \tag{7}$$

Since the variance of $X(t)$ is given by

$$\operatorname{Var}X(t)=m_2(t)+m_1(t)-[m_1(t)]^2\,, \tag{8}$$

the solutions of (4) and (6) enable us to obtain the mean and variance of $X(t)$. Higher moments can be obtained by continued differentiation.

2. – An electron-photon cascade model.

In this section we consider a simple model of an electron-photon cascade. In order to consider a particular process of the Waugh-type, it is necessary to specify the functions $G(t)$ and $h(s,\tau)$ which characterize the process. We now assume

$$G(t)=1-\exp[-\lambda t]\,, \qquad \lambda>0, \tag{9}$$

and

$$(10)\qquad \begin{cases} q_0(\tau) = 1 - \beta \exp[-\alpha\tau], & \alpha > 0, \quad 0 \leqslant \beta \leqslant 1 \\ q_2(\tau) = \gamma \exp[-\alpha\tau], & 0 \leqslant \gamma \leqslant 1 \\ q_1(\tau) = 1 - (q_0(\tau) + q_2(\tau)) = (\beta - \gamma) \exp[-\alpha\tau]. \end{cases}$$

The functions (9) and (10) which characterize the cascade can be described as follows: An electron at absorber depth t has probability $\lambda \Delta t + o(\Delta t)$ of being transformed (experiencing a collision) in the interval $(t, t+\Delta t)$. When transformation takes place the electron can: (i) be absorbed with probability $q_0(\tau)$, where $q_0(\tau)$ is an increasing function of τ, equal to $1-\beta$ when $\tau = 0$, and equal to unity when $\tau = \infty$; (ii) produce a new electron and continue to penetrate the absorber, or be absorbed and produce two new electrons with probability $q_2(\tau)$, where $q_2(\tau)$ is a decreasing function of τ, equal to γ when $\tau = 0$, and equal to zero when $\tau = \infty$; or (iii) continue to penetrate with probability $q_1(\tau)$, where $q_1(\tau)$ is also a decreasing function of τ, equal to $(\beta - \gamma)$ when $\tau = 0$, and equal to zero when $\tau = \infty$. In order for $q_1(\tau)$ to be non-negative we must have $\beta \geqslant \gamma$.

From the above, we see that the possibilities when transformation takes place are that the number of electrons in the cascade can be reduced by one with probability $q_0(\tau)$, unchanged with probability $q_1(\tau)$, or increased by one with probability $q_2(\tau)$. In the definition of the transformation probabilities the variable τ represents *the depth of absorber penetrated by an electron from its formation until it is transformed.* We also remark that the negative-exponential form of $G(t)$ expresses the Markov property of the stochastic cascade.

Before proceeding we add a few remarks concerning the relationship of the above model to some of the stochastic models considered by other workers. Firstly, we note that for parameter values $\alpha = 0$, $\beta = \gamma = 1$, the above model is identical with the pure birth model considered by W. H. FURRY (7). Secondly, for $\alpha = 0$, and $\beta \neq \gamma$, we obtain a model of the birth-and-death type (1, 8). Finally, our model is a generalization of a model considered by N. ARLEY (8) in which the probability of absorption in the interval $(t, t+\Delta t)$ was assumed to be $\mu t\, \Delta t + o(\Delta t)$. For the Arley model, and our model, the total cross-section is a function of τ.

While energy considerations are not explicitly taken into account in our model, the exponential form of the transformation probabilities provides a simple approximation to more realistic cascade processes if one assumes that

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(7) W. H. FURRY: *Phys. Rev.*, **52**, 569 (1957).

(8) N. ARLEY: *On the Theory of Stochastic Processes and Their Applications to the Theory of Cosmic Radiation* (New York, 1949).

the energy dissipated by a particle is proportional to the thickness of absorber traversed since its formation.

From (2), (9) and (10) we obtain the following integral equation for the generating function $\pi(s, t)$:

$$\pi(s,t) = \int_0^t \{ q_0(\tau) + q_1(\tau)\pi(s, t-\tau) + q_2(\tau)\pi^2(s, t-\tau)\} \cdot \lambda \exp[-\lambda\tau]\,d\tau + s\exp[-\lambda t] = \tag{11}$$

$$= \int_0^t \{1 - \beta\exp[-\alpha\tau] + [(\beta-\gamma)\exp[-\alpha\tau]\pi(s, t-\tau) + \gamma\exp[-\alpha\tau]\pi^2(s, t-\tau)\}\lambda\exp[-\lambda\tau]\,d\tau + s\exp[-\lambda t]\,.$$

The above integral equation can be reduced to the differential equation

$$\frac{\partial\pi(s,t)}{\partial t} = \gamma\lambda\pi^2(s,t) - [\alpha + \lambda(1-(\beta-\gamma))]\pi(s,t) + \alpha + \lambda(1-\beta) + \alpha(s-1)\exp[-\lambda t]\,. \tag{12}$$

A solution to this non-linear and non-homogeneous equation which admits a series expansion in s is difficult to obtain. Hence we concern ourselves with the study of the mean and variance of the number of electrons in the cascade.

3. – Integral equations for the moments.

From (4), (9) and (10), we see that the mean number of electrons in the cascade is given by the solution of the integral equation

$$m_1(t) = \int_0^t \lambda(\beta+\gamma)m_1(t-\tau)\exp[-(\alpha+\lambda)\tau]\,d\tau + \exp[-\lambda t]\,. \tag{13}$$

This integral equation can be reduced to the differential equation

$$\frac{dm_1(t)}{dt} = -\{\alpha + \lambda[1-(\beta+\gamma)]\}m_1(t) + \alpha\exp[-\lambda t]\,, \tag{14}$$

the solution of which is

$$(15) \qquad m_1(t) = \frac{1}{[\alpha - \lambda(\beta+\gamma)]}\{\alpha \exp[-\lambda t] - \\ - \lambda(\beta+\gamma)\exp[-[\alpha+\lambda(1-(\beta+\gamma))]t]\}.$$

In solving (14) we have used the initial condition $m_1(0) = 1$, *i.e.* the cascade is initiated by one electron. Should the cascade be initiated by N electrons, which are statistically independent and produce independent cascades, the mean number of electrons at absorber depth t is obtained by multiplying (15) by N.

In connection with the mean number of electrons in the cascade, it is of interest to consider two problems. The first is to determine the mean number of electrons for large absorber depth, *i.e.* as $t \to \infty$. The second is to find the value of t which maximizes the mean, and then obtain the associaed mean number of electrons for various parameter values. For the first problem it is easy to show that as $t \to \infty$ the following limiting values, depending on certain relations between the parameters, obtain:

$$(16) \qquad m_1^* = \lim_{t\to\infty} m_1(t) = \begin{cases} 0, & \text{for } \beta + \leqslant 1, \\ 2, & \text{for } \alpha = \lambda,\ \beta = \gamma = 1, \\ \infty, & \text{for } \alpha < \lambda,\ \beta = \gamma = 1. \end{cases}$$

For the second problem, it is equally easy to show that the value of t which maximizes $m_1(t)$ is

$$(17) \qquad t = t_{\max} = \frac{1}{[\alpha - \lambda(\beta+\gamma)]} \log\left\{\frac{(\beta+\gamma)[\alpha+\lambda(1-(\beta+\gamma))]}{\alpha}\right\}.$$

Since the above expression for $t_{\max}$ can assume a large number of values, depending on the various parameter values, it is of interest to calculate $t_{\max}$ for the special cases considered in the determination of m_1^*. For $\beta+\gamma = 1$ we see that $t_{\max} = 0$; hence the maximum mean value is one (or N in the case $X(0) = N$), and the curve of the mean number of electrons decreases to zero as $t \to \infty$. For $\beta = \gamma$

$$(18) \qquad t_{\max} = \frac{1}{[\alpha - 2\lambda\beta]} \log\left\{\frac{2\beta[\alpha+\lambda(1-2\beta)]}{\alpha}\right\}.$$

In order for $t_{\max}$ to be positive and finite we must have

$$\frac{2\beta[\alpha+\lambda(1-2\beta)]}{\alpha} > 1 \quad \text{and} \quad \alpha - 2\lambda\beta > 0;$$

hence we must have $\lambda < \alpha(2\beta)^{-1}$. In this case $m_1(t)$ attains its maximum at t_{max} given by (18), and then decreases to zero. For $\alpha = \lambda$, $\beta = \gamma = 1$, we find that $t_{max} = \infty$; hence $m_1(t)$ does not achieve a maximum for finite absorber depths. For $t = \infty$, we have from (16) $m_1 = 2$. Finally, for $\alpha < \lambda$, $\beta = \gamma = 1$

$$t_{max} = \frac{1}{\alpha - 2\lambda} \log \left\{ \frac{2(\alpha - \lambda)}{\alpha} \right\}. \tag{19}$$

Since we assume $\alpha < \lambda$, t_{max} for this case does not exist.

In general, if we put $t = t_{max}$ in (15) we obtain

$$m_1(t_{max}) = \frac{\alpha}{[\alpha - \lambda(\beta + \gamma)]} \left\{ \frac{(\beta + \gamma)[\alpha + \lambda(1 - (\beta + \gamma))]}{\alpha} \right\}^{\frac{\lambda}{[\lambda(\beta+\gamma)-\alpha]}} - \frac{\lambda(\beta + \gamma)}{[\alpha - \lambda(\beta + \gamma)]} \left\{ \frac{(\beta + \gamma)[\alpha + \lambda(1 - (\beta + \gamma))]}{\alpha} \right\}^{\frac{\alpha+\lambda(1-(\beta+\gamma))}{[\lambda(\beta+\gamma)-\alpha]}}, \tag{20}$$

as the mean number of electrons at the maximum absorber depth t_{max}.

We now consider the variance of the number of electrons in the cascade. From (6), (9) and (10) we have

$$m_2(t) = 2\gamma\lambda \int_0^t [m_1(t - \tau)]^2 \exp[-(\alpha + \lambda)\tau] \, d\tau + \lambda(\beta + \gamma) \int_0^t m_2(t - \tau) \exp[-(\alpha + \lambda)\tau] \, d\tau. \tag{21}$$

This integral equation can be reduced to the differential equation

$$\frac{dm_2(t)}{dt} = -[\alpha + \lambda(1 - (\beta + \gamma))] m_2(t) + 2\gamma\lambda[m_1(t)]^2. \tag{22}$$

Since $m_1(t)$ is a known function, the above equation can be solved to give $m_2(t)$. The solution of (22) is

$$m_2(t) = 2\gamma\lambda B^{-2} \exp[At] \{\alpha^2(2B + A)^{-1} \exp[(B + A)t] - 2\alpha(B - \alpha)\lambda^{-1} \exp[-\lambda t] + (B - \alpha)^2 A^{-1} \exp[At]\} + C \exp[At], \tag{23}$$

where $A = -[\alpha + \lambda(1 - (\beta + \gamma))]$, $B = \alpha - \lambda(\beta + \gamma)$, and C is the constant of integration. Using the initial condition $m_2(0) = 0$ we find

$$C = -2\gamma\lambda B^{-2} \{\alpha^2(2B + A)^{-1} - 2\alpha(B - \alpha)\lambda^{-1} + (B - \alpha)^2 A^{-1}\}.$$

From (8), (15) and (23), we obtain the rather complicated expression for the variance of the number of electrons in the cascade:

$$\text{(24)}\qquad \operatorname{Var} X(t) = B^{-2}\{[2\gamma\lambda(2B+A)^{-1}-1]\alpha^2\exp[-2\lambda t] + B\exp[-\lambda t] - \\ - 2\alpha(2\gamma+1)(B-\alpha)\exp[(2A+B)t] + (2\gamma\lambda A^{-1}-1)(B-\alpha)^2\exp[2At] + \\ + [B(B-\alpha) - 2\gamma\lambda[\alpha^2(2B+A)^{-1} + 2\alpha(B-\alpha)\lambda^{-1} + (B-\alpha)^2A^{-1}]]\exp[At]\}\,.$$

If we consider the asymptotic variance for large absorber depths, we find

$$\text{(25)}\qquad \lim_{t\to\infty}\operatorname{Var} X(t) = \begin{cases} 0\,, & \text{for } \beta+\gamma<1 \\ \infty\,, & \text{for } \alpha<\lambda,\ \ \beta=\gamma=1\,. \end{cases}$$

For the case $\alpha=\lambda$, and $\beta=\gamma=1$, $A=0$ and $B<0$; hence in order to obtain the variance in this case it is necessary to determine $m_1(t)$ and $m_2(t)$ when the above relations between the parameters obtain. From equation (22) we obtain

$$\text{(26)}\qquad \frac{\mathrm{d}m_2(t)}{\mathrm{d}t} = 2\lambda[m_1(t)]^2,$$

where

$$\text{(27)}\qquad m_1(t) = 2 - \exp[-\lambda t].$$

Solving (26) with the initial condition $m_2(0)=0$, we obtain

$$\text{(28)}\qquad m_2(t) = 8(\lambda t + \exp[-\lambda t]) - \exp[-2\lambda t] - 7\,.$$

Hence

$$\text{(29)}\qquad \lim_{t\to\infty}\operatorname{Var} X(t) = \infty\,,$$

for $\alpha=\lambda$, $\beta=\gamma=1$.

4. – Coefficient of variation or relative fluctuation.

We now add a few remarks on the coefficient of variation (or relative fluctuation) of the number of electrons in the cascade. The importance of this function has been stressed by several authors ([8]). The coefficient of variation is defined as

$$\text{(30)}\qquad V(t) = \frac{(\operatorname{Var} X(t))^{\frac{1}{2}}}{\mathcal{E}X(t)}\,.$$

From (8) we see that

$$V(t) = \frac{m_2(t)}{[m_1(t)]^2} + \frac{1}{m_1}(t) - 1 . \tag{31}$$

Because the expression for $m_2(t)$ is rather complicated, the function $V(t)$ is difficult to discuss for finite absorber depths. However the asymptotic coefficient of variation $V^* = \lim V(t)$ can be obtained from (16), (25) and (29). From a study of certain special cases we observe that the introduction of transformation probabilities depending on absorber depth increases the fluctuation.

5. – Probability of cascade termination.

Since we have introduced a probability of absorption, $q_0(\tau)$, it is of interest to determine the probability, say $P(t)$, that the cascade will terminate at absorber depth t, $t < 0$, *i.e.* $P(t) = \Pr(X(t) = 0)$. It has been shown (6) that $P(t)$ is a solution of the integral equation,

$$P(t) = \int_0^t \{q_0(\tau) + q_1(\tau)P(t-\tau) + q_2(\tau)P^2(t-\tau)\}\, \mathrm{d}G(\tau) = \tag{32}$$

$$= \int_0^t \{1 - \beta \exp[-\alpha\tau] + (\beta - \gamma)\exp[-\alpha\tau]P(t-\tau) +$$

$$+ \gamma \exp[-\alpha\tau]P^2(t-\tau)\}\lambda \exp[-\lambda\tau]\,\mathrm{d}\tau .$$

Like equations (11), (13) and (21), the above integral equation can be reduced to a differential equation and solved to give $P(t)$. However, in order to determine the probability that the cascade will eventually terminate, we have only to consider an algebric equation.

From the theory of discrete branching processes ([5]) it is known that the probability of eventual termination is given by the smallest non-negative solution of the functional equation $\pi(s) = s$. For age-dependent processes it has been shown ([2]) that this probability, say $Q = \lim_{t\to\infty} P(t)$, is given by the unique non-negative root, less than unity, of $h(Q) = Q$. For the Waugh process it is necessary to consider, in place of the above equation, the functional equation

$$h(Q, \tau) = Q . \tag{33}$$

Hence, for our particular model, we seek the solution of

$$h(Q, \tau) = A + BQ + CQ^2 = Q \,, \tag{34}$$

where

$$A = \int_0^\infty q_0(\tau) g(\tau)\, \mathrm{d}\tau = \int_0^\infty \lambda[1 - \beta \exp[-\alpha\tau]] \exp[-\lambda\tau]\, \mathrm{d}\tau = 1 - \frac{\beta\lambda}{\alpha + \lambda} \,,$$

$$B = \int_0^\infty q_1(\tau) g(\tau)\, \mathrm{d}\tau = \int_0^\infty \lambda(\beta - \gamma) \exp[-(\alpha + \lambda)\tau]\, \mathrm{d}\tau = \frac{(\beta - \gamma)\lambda}{\alpha + \lambda} \,,$$

$$C = \int_0^\infty q_2(\tau) g(\tau)\, \mathrm{d}\tau = \int_0^\infty \gamma\lambda \exp[-(\alpha + \lambda)\tau]\, \mathrm{d}\tau = \frac{\gamma\lambda}{\alpha + \lambda} \,,$$

with $A + B + C = 1$. Therefore, the probability that the cascade will eventually terminate is given by the smallest non-negative solution, less than unity, of

$$\gamma\lambda Q^2 - [\alpha - \lambda(1 - (\beta + \gamma))]Q + \alpha + \lambda(1 - \beta) = 0 \,. \tag{35}$$

For values such that

$$[\alpha - \lambda(1 - (\beta + \gamma))]^2 > 4\gamma\lambda(\alpha + \lambda(1 - \beta))$$

two real roots, say ζ_1 and ζ_2, of (35) exist; and $Q = \min(\zeta_1, \zeta_2)$.

RIASSUNTO (*)

Nel presente lavoro si considera un modello semplice di una cascata di elettroni e fotoni in cui le probabilità di trasformazione sono funzioni dello spessore dell'assorbitore. Si ricavano la media e la varianza del numero degli elettroni nella cascata in funzione dello spessore dell'assorbitore e se ne discutono le proprietà. Si dà anche un'equazione che fornisce la probabilità che la cascata si arresti.

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Analytic Properties of Scattering Amplitudes as Functions of Momentum Transfer.

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Summary. — Scattering amplitudes are shown to have analytic properties as functions of momentum transfer. The partial wave expansions which define physical scattering amplitudes continue to converge for complex values of the scattering angle, and define uniquely the amplitudes appearing in the unphysical region of non-forward dispersion relations. The expansions converge for all values of momentum transfer for which dispersion relations have been proved.

1. – Introduction.

The purpose of this note is to derive some properties of scattering amplitudes which follow from causality in relativistic quantum theory. It will be shown that a scattering amplitude has—for fixed energy—analytic properties as a function of scattering angle or momentum transfer. This consequence of causality is distinct from the existence of dispersion relations [1-4] which express analytic properties of a scattering amplitude as a function of energy for fixed momentum transfer. However, our results are of interest mainly in connection with dispersion relations for non-forward scattering. They imply that for all values of momentum transfer for which dispersion relations have

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(1) M. L. Goldberger: *Phys. Rev.*, **99**, 979 (1955).

(2) K. Symanzik: *Phys. Rev.*, **100**, 743 (1957).

(3) N. Bogoliubov, B. Medvedev and M. Polivanov: *lecture notes*. Translated at the Institute for Advanced Study, (Princeton, 1957).

(4) H. J. Bremermann, R. Oehme and J. G. Taylor: *Phys. Rev.*, **109**, 2178 (1958). These papers contain numerous other references.

been established ([3,4])—in π-N scattering this is the case for momentum transfer $< (\frac{2}{3}\cdot(2m+\mu)/(2m-\mu))^{\frac{1}{2}}\cdot 4\mu$—the so-called non-physical region is completely determined by the physical phase shifts via the partial wave expansion. This possibility has been suggested previously ([5]).

We disregard spins and consider the scattering of a charged particle of mass m (nucleon) with initial momentum p, final momentum p', and a neutral particle of mass $\mu < 2m$ (meson) with initial momentum k, final momentum k'. We assume in addition that these particles are not coupled to other particles of charge zero and mass $< 2\mu$ or of charge one and mass $< m+\mu$. Our results are valid also for the scattering of equal particles ($m = \mu$) under corresponding restrictions on the mass spectrum.

As is well known, several equivalent expressions for the scattering matrix in terms of Heisenberg operators may be given. With the notation

$$\langle p'k' \text{ out} | pk \text{ in}\rangle = \langle p'k' \text{ in} | pk \text{ in}\rangle + i(2\pi)\,\delta(p+k-p'-k')T$$

the amplitude T may be written as ([6])

$$T = -\int \mathrm{d}^4x \exp\left[\frac{i(k+k')}{2}x\right]\langle p'|R'A\left(\frac{x}{2}\right)A\left(-\frac{x}{2}\right)|p\rangle\,, \tag{1}$$

or

$$T = -\int \mathrm{d}^4x \exp\left[\frac{i(k'-p')}{2}x\right]\langle 0|R'A\left(\frac{x}{2}\right)\psi\left(-\frac{x}{2}\right)|pk \text{ in}\rangle\,. \tag{2}$$

$A(x)$ and $\psi(x)$ are the meson and nucleon field operators. R' denotes a retarded commutator. For example,

$$R'A(x)\psi(y) = -i(\Box x-\mu^2)(\Box y - m^2)\,\theta(x-y)[A(x),\psi(y)]\,.$$

The state vectors refer to incoming or outgoing particles with definite momenta as indicated.

Eq. (1) is used in the derivation of dispersion relations. Eq. (2) on the other hand yields directly information about the scattering amplitude as a function of momentum transfer, since—in the center of mass system—this variable appears only in the exponential. However, more information is obtained in both cases by observing that from either (1) or (2) the following

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([5]) M. L. Goldberger: *Proceedings of the Sixth Annual Rochester Conference*, (New York, 1956).

([6]) H. Lehmann, K. Symanzik and W. Zimmermann, *Nuovo Cimento*, **6**, 319 (1957).

expression for the imaginary part of the amplitude may be derived,

$$(3)\quad \operatorname{Im} T = \pi \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2 \exp\left[\frac{i(k'-p')x_1}{2} - \frac{i(k-p)x_2}{2}\right] \cdot \sum_\gamma \langle 0|R'A\left(\frac{x_1}{2}\right)\psi\left(-\frac{x_1}{2}\right)|p+k,\gamma\rangle \cdot \langle p+k,\gamma|R'A\left(\frac{x_2}{2}\right)\psi^+\left(-\frac{x_2}{2}\right)|0\rangle ,$$

$\sum_\gamma$ denotes a sum over all states with total four-momentum $p+k$. Eq. (3) is (on the energy shell) equivalent to the unitary requirement for the scattering amplitude. It was first used by BOGOLIUBOV *et al.* [3]. A simple proof of this relation is given in Sect. **3**.

The general method used to obtain explicit consequences of these expressions may be described as follows:
In each case the scattering amplitude—or its imaginary part—appears as the Fourier transform of a retarded commutator or of a sum over products of such commutators. Therefore it is simply related to the Fourier-transform of the corresponding unretarded commutator which is in the case of (2) or (3) given by

$$(4)\quad F(q) = \int \mathrm{d}^4x \exp\,[iqx]\langle 0|\left[j\left(\frac{x}{2}\right), f\left(-\frac{x}{2}\right)\right]|p+k,\gamma\rangle ,$$

where $j(x) = (\Box - \mu^2)A(x)$; $f(x) = (\Box - m^2)\psi(x)$. We know about $F(q)$:

(4*a*) it is the Fourier transform of a function that vanishes for space-like x;

$F(q)$ vanishes unless

$$(4b)\quad \begin{cases} \dfrac{p_0+k_0}{2} + q_0 \geqslant 0 \quad \text{and} \quad \left(\dfrac{p+k}{2}+q\right)^2 \geqslant m_1^2, \\ \text{or} \\ \dfrac{p_0+k_0}{2} - q_0 \geqslant 0 \quad \text{and} \quad \left(\dfrac{p+k}{2}-q\right)^2 \geqslant m_2^2. \end{cases}$$

The latter statement follows directly if a sum over intermediate states is introduced in (4). m_1 and m_2 are the masses of the lowest intermediate states which contribute to the two terms of the commutator. In the π-N case $m_1 = 3\mu$, $m_2 = m+\mu$.

DYSON (7) has solved the problem of finding a representation of all functions satisfying these conditions. His result is: For $F(q)$ to satisfy (4*a*) and (4*b*) it is necessary and sufficient that it can be represented as

$$F(q) = \int \mathrm{d}^4 u \int_0^\infty \mathrm{d}\varkappa^2 \varepsilon(q_0 - u_0)\delta[(q-u)^2 - \varkappa^2]\varphi(u, \varkappa^2)\,. \tag{5}$$

$\varphi(u, \varkappa^2)$ is arbitrary if the vectors $(p+k)/2+u$ and $(p+k)/2-u$ both lie in the forward light-cone and

$$\varkappa \geqslant \mathrm{Max}\left\{0;\ m_1 - \sqrt{\left(\frac{p+k}{2}+u\right)^2};\ m_2 - \sqrt{\left(\frac{p+k}{2}-u\right)^2}\right\}.$$

φ vanishes outside this region. It depends, of course, also on the quantum numbers γ and on $p+k$. All our results will be based on applications of Dyson's theorem.

For the Fourier transform of the retarded commutator which appears in (2) or (3) we have the relation $(q' = (q_0', \boldsymbol{q}))$

$$F_R(q) = -\frac{1}{2\pi}\int \frac{\mathrm{d}q_0' F(q')}{q_0' - q_0}\,; \qquad \mathrm{Im}\, q_0 \geqslant 0\,,$$

if F_R is sufficiently bounded.

Inserting (5) gives

$$F_R(q) = -\frac{1}{2\pi}\int \mathrm{d}^4 u \int \frac{\mathrm{d}\varkappa^2 \varphi(u, \varkappa^2)}{(q-u)^2 - \varkappa^2}\,. \tag{6}$$

In general we cannot expect F_R to be bounded enough for (6) to hold in this form. The necessary modification (8) does not alter the analytic properties we are interested in. It is therefore sufficient to discuss (6). We shall use it both for Eq. (2) and Eq. (3).

2. – Momentum transfer properties of scattering amplitudes.

By inserting (6) into Eq. (2) we obtain

$$T = \frac{1}{2\pi}\int \frac{\mathrm{d}^4 u\, \mathrm{d}\varkappa^2 \varphi(u, \varkappa^2, p, k)}{((k'-p')/2 - u)^2 - \varkappa^2}\,, \tag{7}$$

(7) F. J. DYSON: *Phys. Rev.*, **110**, 1460 (1958).

(8) R. JOST and H. LEHMANN: *Nuovo Cimento*, **5**, 1598 (1957), Eq. (4.5).

φ is an invariant function of the vectors u, p, k. The integration extends over the region given in (5). We choose the center of mass system to evaluate (7) and introduce the variables

$$(8)\qquad \begin{cases} W^2 = (p+k)^2; \qquad \Delta^2 = -\dfrac{(k-k')^2}{4}, \\ \text{or} \\ K^2 = \dfrac{[W^2-(m+\mu)^2][W^2-(m-\mu)^2]}{4W^2}; \qquad \cos\vartheta = 1 - \dfrac{2\Delta^2}{K^2}. \end{cases}$$

φ depends then only on u^2, u_0, $\boldsymbol{u}\cdot\boldsymbol{k}$, $\varkappa^2$, W. It vanishes outside

$$(9)\qquad \begin{cases} 0 \leqslant u \leqslant W/2; \quad -W/2 + u \leqslant u_0 \leqslant W/2 - u, \\ \varkappa \geqslant \mathrm{Max}\left\{0;\, m_1 - \sqrt{(W/2+u_0)^2 - u^2};\, m_2 - \sqrt{(W/2-u_0)^2-u^2}\right\}. \end{cases}$$

Introducing polar co-ordinates in $\boldsymbol{u}$-space, (7) becomes:

$$(10)\qquad T(W,\cos\vartheta) = -\frac{1}{4\pi K}\int \mathrm{d}u_0 \int u\,\mathrm{d}u \int \mathrm{d}\varkappa^2 \int_0^{2\pi}\mathrm{d}\alpha \int_0^{\pi}\mathrm{d}\beta \cdot \cdot \frac{\varphi(u_0, u^2, \cos\alpha\sin\beta, \varkappa^2, W)}{\dfrac{K^2+u^2+\varkappa^2-(u_0+(m^2-\mu^2)/2W)^2}{2Ku\sin\beta} - \cos(\vartheta-\alpha)},$$

or

$$(11)\qquad T(W,\cos\vartheta) = \int_{x_0(W)}^{\infty}\mathrm{d}x \int_0^{2\pi}\mathrm{d}\alpha \frac{\bar{\varphi}(x,\cos\alpha, W)}{x-\cos(\vartheta-\alpha)},$$

with

$$\bar{\varphi}(x,\cos\alpha,W) = -\frac{1}{4\pi K}\int \mathrm{d}u_0 \int u\,\mathrm{d}u \int \mathrm{d}\varkappa^2 \int_0^{\pi}\mathrm{d}\beta \cdot \cdot \delta\left[x - \frac{K^2+u^2+\varkappa^2-(u_0+(m^2-\mu^2)/2W)^2}{2Ku\sin\beta}\right] \cdot \varphi(u_0, u^2, \cos\alpha\sin\beta, \varkappa^2, W).$$

The lower limit $x_0(W)$ is determined by

$$x_0(W) = \mathrm{Min}\left\{\frac{K^2+u^2+\varkappa^2-(u_0+(m^2-\mu^2)/2W)^2}{2Ku}\right\},$$

if u_0, u, $\varkappa$ vary over the region (9). The minimum can be calculated in an elementary manner. The result is

$$x_0(W) = \left[1 + \frac{(m_1^2 - \mu^2)(m_2^2 - m^2)}{K^2[W^2 - (m_1 - m_2)^2]}\right]^{\frac{1}{2}} \tag{12}$$

In (11) the scattering angle $\cos\vartheta$ appears only in the denominator. We may therefore consider $\cos\vartheta$ as a complex variable and the scattering amplitude as an analytic function of $\cos\vartheta$. Moreover, this can be done separately for the real and imaginary parts of the amplitude.

Singularities of these functions can occur only if the denominator on the right hand side of (11) vanishes. That is for

$$\cos\vartheta = x \cdot \cos\alpha \pm i\sqrt{x^2 - 1}\,\sin\alpha\,.$$

We have therefore the following result:

(13) The real part and the imaginary part of the scattering amplitude are analytic functions of $\cos\vartheta$, regular inside an ellipse in the $\cos\vartheta$-plane with center at the origin and with axes x_0, $\sqrt{x_0^2 - 1}$.

We shall see presently—making use of Eq. (3)—that the imaginary part of the amplitude is regular in a larger domain, namely:

(14) $\mathrm{Im}\,T(W, \cos\vartheta)$ is regular in $\cos\vartheta$ inside an ellipse with center at the origin and with axes $2x_0^2 - 1$; $2x_0 \cdot \sqrt{x_0^2 - 1}$. x_0 is given by (12).

Using $\cos\vartheta = 1 - (2\Delta^2/K^2)$ we can, of course, re-express (13) and (14) as analytic properties of the scattering amplitude as a function of momentum transfer.

These results (we defer the proof of (14)) lead—using well-known mathematical theorems (9)—to the following properties of the partial wave expansion of the scattering amplitude. Let

$$\left\{\begin{aligned} &T(W, \cos\vartheta) = \frac{1}{\pi^2}\frac{W}{K}\sum_{l=0}^{\infty}(2l+1)C_l(W)\,P_l(\cos\vartheta)\,, \\ &\text{with}\quad C_l(W) = \frac{\pi^2}{2}\frac{K}{W}\int_{-1}^{1}\mathrm{d}\cos\vartheta\, T(W, \cos\vartheta)\,P_l(\cos\vartheta)\,. \end{aligned}\right. \tag{15}$$

(9) E. T. Whittaker and G. N. Watson. *A course of modern analysis*, 4th Ed., (Cambridge 1940), p. 322; G. Szegö. *Orthogonal Polynomials*, (New York, 1939), p. 238.

The Legendre series converges inside the domain of regularity of the represented functions; *i.e.* for $\cos\vartheta$ inside the ellipses (13) or (14) for Re T or Im T respectively. Also

$$\overline{\lim_{l\to\infty}} |\operatorname{Re} C_l(W)|^{1/l} \leqslant \frac{1}{x_0 + \sqrt{x_0^2 - 1}}, \tag{13a}$$

$$\overline{\lim_{l\to\infty}} |\operatorname{Im} C_l(W)|^{1/l} \leqslant \frac{1}{(x_0 + \sqrt{x_0^2 - 1})^2}. \tag{14a}$$

Taking into account the unitarity relation

$$\operatorname{Im} C_l(W) \geqslant [\operatorname{Re} C_l(W)]^2 + [\operatorname{Im} C_l(W)]^2, \tag{16}$$

we may note that (13*a*) is actually a consequence of (14*a*); *i.e.* if Im $T(W, \cos\vartheta)$ is regular in the domain (14) it follows immediately that Re $T(W, \cos\vartheta)$ is regular in (13).

We cannot conclude, of course, that the amplitude $T(W, \cos\vartheta)$ actually has singularities on the boundary of the domains (13) or (14). Using more physical information it may well be possible to improve these results.

To discuss the connection of the above statements with the non-physical region of dispersion relations, let us consider Im $T(K^2, \Delta^2)$, the imaginary part of the amplitude, as a function of c.m. momentum and momentum transfer—the physical region is given by $K^2 > \Delta^2$. However, in the dispersion relation Im $T(K^2, \Delta^2)$ is needed for all $K^2 > 0$. The expansion

$$\operatorname{Im} T(K^2, \Delta^2) = \frac{1}{\pi^2} \frac{W}{K} \sum_{l=0}^{\infty} (2l+1) \operatorname{Im} C_l(W) P_l\left(1 - \frac{2\Delta^2}{K^2}\right), \tag{15a}$$

defines a continuation of Im $T(K^2, \Delta^2)$ into the non-physical region. The series converges if

$$\Delta^2 < K^2 x_0^2 = K^2 + \frac{(m_1^2 - \mu^2)(m_2^2 - m^2)}{W^2 - (m_1 - m_2)^2};$$

i.e. it converges for all $K^2 > 0$ provided

$$\Delta^2 < \operatorname{Min}\{K^2 x_0^2\}.$$

This leads to the restriction

$$\Delta^2 < \frac{8}{3} \frac{2m + \mu}{2m - \mu} \cdot \mu^2 \approx 3\mu^2,$$

in the π-N case. ($\Delta^2 < 2\mu^2$ for equal particle scattering with $m_1 = m_2 = 2\mu$).

We have still to show that the continuation given by (15a) is indeed the correct definition of the non-physical region in the dispersion relation.

3. – Connection with dispersion relations.

To identify Im $T(K^2, \Delta^2)$ as given by (15a) with the dispersion relation integrand it is convenient to consider (1) (which defines a function T for arbitrary real vectors k, k') not only on the energy shell ($k^2 = k'^2 = \mu^2$) but for the more general case $k^2 = k'^2 = \zeta$; keeping $p^2 = p'^2 = m^2$. T can then be considered as a function of

$$\omega = \frac{(k+k')(p+p')}{2\sqrt{(p+p')^2}} \qquad \zeta = k^2 = k'^2, \qquad \Delta^2 = -\frac{(p-p')^2}{4}.$$

We derive first Eq. (3).

(1) leads directly to

$$\text{(17)} \quad \text{Im } T(w, \zeta, \Delta^2) = \frac{1}{2}\int \mathrm{d}^4x \exp\left[i\left(\frac{k+k'}{2}\right)x\right]\langle p'|\left[j\left(\frac{x}{2}\right), j\left(-\frac{x}{2}\right)\right]|p\rangle = \\ = \frac{1}{2}\{M(\omega, \zeta, \Delta^2) - M(-\omega, \zeta, \Delta^2)\},$$

with

$$\text{(18)} \quad M(\omega, \zeta, \Delta^2) = \int \mathrm{d}^4x \exp\left[i\left(\frac{k+k'}{2}\right)x\right]\langle p'|j\left(\frac{x}{2}\right), j\left(-\frac{x}{2}\right)|p\rangle = \\ = (2\pi)^4 \sum_\gamma \langle p'|j(0)|p+k, \gamma\rangle\langle p+k, \gamma|j(0)|p\rangle .$$

Let $\varphi\,\text{in}\,(p')$ denote the annihilation operator for an incoming nucleon with momentum p'. Then

$$\langle p'|j(0)|p+k, \gamma\rangle = \langle 0|\varphi\,\text{in}\,(p')j(0)|p+k, \gamma\rangle = \langle 0|[\varphi\,\text{in}\,(p'), j(0)]|p+k, \gamma\rangle$$

$$\text{if} \quad (p+k-p')^2 = k'^2 = \zeta < 9\mu^2$$

since $\langle 0|j(0)\varphi\,\text{in}\,(p')|p+k, \gamma\rangle = 0$ in this case. With the relation [5]

$$[\varphi\,\text{in}\,(p'), j(0)] = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^4x \exp[ip'x]R'A(0)\psi(x),$$

and an analogous treatment of the second factor in (18) we have

$$(19)\quad M(\omega,\zeta,\Delta^2)=2\pi\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\exp\left[ip'x_1-ipx_2\right]\sum_\gamma\langle 0|R'A(0)\psi(x_1)|p+k,\gamma\rangle\cdot$$

$$\cdot\langle p+k,\gamma|R'A(0)\psi^+(x_2)|0\rangle=2\pi\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\exp\left[i\left(\frac{k'-p'}{2}\right)x_1-i\left(\frac{k-p}{2}\right)x_2\right]\cdot$$

$$\cdot\sum_\gamma\langle 0|R'A\left(\frac{x_1}{2}\right)\psi\left(-\frac{x_1}{2}\right)|p+k,\gamma\rangle\cdot\langle p+k,\gamma|R'A\left(\frac{x_2}{2}\right)\psi^+\left(-\frac{x_2}{2}\right)|0\rangle\,.$$

We note that the imaginary part of the physical scattering amplitude is given by

$$(20)\qquad \operatorname{Im}T=\tfrac{1}{2}M(\omega,\mu^2,\Delta^2)\,.$$

The term $M(-\omega,\mu^2,\Delta^2)$ does not contribute since it vanishes for $\omega\geqslant\sqrt{\Delta^2+\mu^2}$. Hence (19) is on the energy shell equivalent to (3).

To find analytic properties of M as given by (19) we use again the integral representation (6). This leads to

$$(21)\quad M(\omega,\zeta,\Delta^2)=\frac{1}{2\pi}\int\frac{\mathrm{d}^4u_1\,\mathrm{d}\varkappa_1^2\,\mathrm{d}^4u_2\,\mathrm{d}\varkappa_2^2\Phi(u_1,\varkappa_1,u_2,\varkappa_2,p+k)}{\left[((k'-p')/2-u_1)^2-\varkappa_1^2\right]\left[((k-p)/2-u_2)^2-\varkappa_2^2\right]}\,,$$

where

$$\Phi(u_1,\varkappa_1,u_2,\varkappa_2,p+k)=\sum_\gamma\varphi_\gamma(u_1,\varkappa_1,p+k)\,\varphi_\gamma^*(u_2,\varkappa_2,p+k)$$

is built up from the weight functions φ_γ corresponding to the individual terms on the right hand side of (19). Φ is a real, invariant function which satisfies the support conditions (6) in each pair of variables u, $\varkappa$ separately. We choose the center of mass system and replace ω by

$$W^2=2\omega\sqrt{\Delta^2+m^2}+2\Delta^2+m^2+\zeta\,.$$

Then

$$(22)\qquad M(\omega,\zeta,\Delta^2)=\frac{1}{2\pi}\cdot$$

$$\cdot\int\frac{\mathrm{d}^4u_1\,\mathrm{d}\varkappa_1^2\,\mathrm{d}^4u_2\,\mathrm{d}\varkappa_2^2\Phi(u_{10},u_1^2,\varkappa_1^2,u_{20},u_2^2,\varkappa_2^2,\boldsymbol{u}_1\boldsymbol{u}_2/u_1u_2,W)}{\left[((m^2-\zeta)/2W+u_{10})^2-\varkappa_1^2-(\boldsymbol{k}'-\boldsymbol{u}_1)^2\right]\left[((m^2-\zeta)/2W+u_{20})^2-\varkappa_2^2-(\boldsymbol{k}-\boldsymbol{u}_2)^2\right]}\cdot$$

With polar co-ordinates

$$(23)\qquad M(W,\zeta,\Delta^2)\,\frac{1}{8\pi\,K^2(\zeta)}\int\mathrm{d}u_0\,u_i\,\mathrm{d}u_i\,\mathrm{d}\varkappa_i^2\int_0^{2\pi}\mathrm{d}\alpha\int_0^{\pi}\mathrm{d}\beta_1\int_0^{\pi}\mathrm{d}\beta_2\int_0^{2\pi}\mathrm{d}\chi\cdot$$

$$\cdot\frac{\Phi(u_0,u_i^2,\varkappa_i^2,\cos\alpha\sin\beta_1\sin\beta_2+\cos\beta_1\cos\beta_2,W)}{\left[x_1(\zeta)-\cos(\vartheta-\chi)\right]\left[x_2(\zeta)-\cos(\chi-\alpha)\right]}\,,$$

$$x_i(\zeta) = \frac{K^2(\zeta) + u_i^2 + \varkappa_i^2 - ((m^2 - \zeta)/2W + u_{i0})^2}{2K(\zeta)u_i \sin \beta_i};$$

$$K^2(\zeta) = \frac{(W^2 + m^2 - \zeta)^2 - 4m^2W^2}{4W^2}.$$

We note that

$$\int_0^{2\pi} \mathrm{d}\chi \frac{1}{[x_1 - \cos(\vartheta - \chi)]} \cdot \frac{1}{[x_2 - \cos(\chi - \alpha)]} =$$

$$= 2\pi \frac{(x_1/(\sqrt{x_1^2 - 1})) + (x_2/(\sqrt{x_2^2 - 1})}{(x_1x_2 + \sqrt{x_1^2 - 1}\sqrt{x_2^2 - 1} - \cos(\vartheta - \alpha)}.$$

From (23) analytic properties in ζ and Δ^2 follow. On the energy shell, *i.e.* $\zeta = \mu^2$, we can introduce $y = x_1x_2 + \sqrt{x_1^2 - 1}\sqrt{x_2^2 - 1}$ as a new integration variable. Only integrations over y and α remain; the other integrations result only in a new weight function $\bar{\Phi}(y, \cos\alpha, W)$.

The minimum value of y is

$$y_0 = \mathrm{Min}\{x_1x_2 + \sqrt{x_1^2 - 1}\sqrt{x_2^2 - 1}\} = 2x_0^2 - 1.$$

Therefore

$$M(W^2, \Delta^2) = 2 \operatorname{Im} T = \int_{2x_0^2 - 1}^{\infty} \mathrm{d}y \int_0^{2\pi} \mathrm{d}\alpha \frac{\bar{\Phi}(y, \cos\alpha, W)}{y - \cos(\vartheta - \alpha)}.$$

This proves the statement (14).

It can be seen now that the analytic continuation of Im T defined by (15*a*) yields the correct non-physical part of the dispersion relation integrand. In the proofs of these relations ([3,4]) it is shown first that—as a consequence of Eq. (1)—a dispersion relation in ω holds if ζ is taken real and $\zeta < -\Delta^2$. The absorptive part in this relation is $M(W, \zeta, \Delta^2)$ as given by (18) and (23). The dispersion relation for the physical value $\zeta = \mu^2$ is then obtained by analytic continuation in ζ, provided $M(W, \zeta, \Delta^2)$ is an analytic function of ζ regular for Re $\zeta \leqslant \mu^2$ in a neighborhood of the real axis. It follows from (23) that this condition is satisfied if

$$\Delta^2 < \mathrm{Min}\{K^2 x_0^2\}.$$

This is also the condition for the convergence of the Legendre series. The absorptive part of the dispersion relation is then given by (23) with $\zeta = \mu^2$; *i.e.* the non-physical region is obtained by analytic continuation in Δ^2 which can be carried out by the Legendre expression (15*a*).

The possibility of evaluating the non-physical region in this manner has been discussed earlier (5). While no proofs were given, it was believed on the basis of threshold arguments that such a procedure could be valid only if $\Delta^2 < \mu^2$, due to a branch point of the scattering amplitude as a function of Δ^2. We have shown that the expansion converges also for higher values of Δ^2; the limit being $\Delta^2 = 2\mu^2$ in the case of equal particle scattering. We believe that this is due to the fact that the real and imaginary part of the amplitude are separately analytic functions of Δ^2 and have different properties. For the dispersion relation only the imaginary part is needed and it has a larger domain of regularity. The mentioned branch point is likely to be present in the real part.

While we have no good reason to believe that our results are best possible, the expected appearance of a singularity in the real part gives us—via the unitarity relation (16)—an upper limit to the values of Δ^2 for which the Legendre expansion for the imaginary part might converge. In the case of equal particle scattering the expected branch point of the real part at $\Delta^2 = W^2/4$ leads to the limitation $\Delta^2 < 8\mu^2$ for the Legendre expansion of the imaginary part.

* * *

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RIASSUNTO (*)

Si dimostra che le ampiezze di scattering hanno proprietà analitiche come funzioni del trasferimento dei momenti. Gli sviluppi parziali delle funzioni d'onda che definiscono le ampiezze fisiche di scattering continuano a convergere per valori complessi dell'angolo di scattering e definiscono unicamente le ampiezze che compaiono nella regione non fisica delle relazioni di dispersione non in avanti. Gli sviluppi convergono per tutti i valori del trasferimento dei momenti per cui sono state dimostrate esatte le relazioni di dispersione.

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The (γ, n) Reaction in ^{4}He Nuclei.

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(ricevuto il 31 Luglio 1958)

Summary. — The photographic plate method has been used to detect ^{3}He particles emitted from helium gas exposed to bremsstrahlung of maximum energy 70 MeV. The ^{4}He$(\gamma, n)^3$He cross-section is found to vary between values of (0.63 ± 0.10) mb at 40 MeV and (0.25 ± 0.04) mb at 60 MeV, the integrated cross-section being 7.5 MeV mb between these limits. The angular distribution of the ^{3}He particles in the C.M. system has a peak in the forward direction and may be described by the expression $f(\theta) = B \sin^2\theta (1 + C \cos\theta)$ with C approximately equal to 0.5.

1. – Introduction.

The photodisintegration of ^{4}He nuclei by bremsstrahlung of high energy was observed in helium-filled cloud chambers by Gaerttner and Yeater [1] and by Nicolai and Goldwasser [2]. Their results indicated that the integrated cross-sections for the (γ, n) process are rather greater than the corresponding values for the (γ, p) process, but detailed excitation functions were not obtained. The photoneutrons from helium were detected by Ferguson, Halpern, Nathans and Yergin [3], who showed that the (γ, n) cross-

(1) E. R. Gaerttner and M. L. Yeater: *Phys. Rev.*, **83**, 146 (1951).
(2) V. O. Nicolai and E. L. Goldwasser: *Phys. Rev.*, **94**, 755A (1954).
(3) G. A. Ferguson, J. Halpern, R. Nathans and P. F. Yergin: *Phys. Rev.*, **95**, 776 (1954).

section increases from zero at the threshold energy of 20.6 MeV to a value of about 1.3 millibarns near 25 MeV, higher energies being unattainable. Experiments by FULLER (4) with photographic plates arranged to record photoprotons emitted from helium gas yielded a (γ, p) cross-section curve rising to a maximum value of 1.8 millibarns near 26 MeV and falling sharply to less than 0.1 millibarn at 40 MeV. A curve of similar shape for the same reaction was obtained by REID, SWINBANK and ATKINSON (5) using a cloud chamber. DE SAUSSURE and OSBORNE (6) used photoplates to record the ^3He particles emitted in the (γ, n) reaction and found a (γ, n) cross-section falling slowly from 0.5 millibarn at 40 MeV to 0.1 millibarn at 120 MeV. Despite the fact that the (γ, p) and (γ, n) processes are essentially similar in ^4He there appeared to be wide discrepancies between the two cross-sections, especially near 40 MeV.

More recently a cloud chamber investigation of the photodisintegration reactions in ^4He has been reported by GORBUNOV and SPIRIDONOV (7) in a paper giving detailed results for the (γ, p) reaction. Before this work became known, the present authors had begun to scan photoplates which had been exposed in helium irradiated by bremsstrahlung from the 70 MeV beta-synchrotron at Queen's University, Kingston, Ontario. The preliminary results gave information about the (γ, n) cross-section above 40 MeV which was reported to the Conference on Photonuclear Reactions at Washington, D.C., in April 1958. More detailed studies have been pursued in order to compare the data with the (γ, p) cross-section curve published by GORBUNOV and SPIRIDONOV. Particular attention has been paid to the (γ, n) cross-section between 40 MeV and 60 MeV and to the angular distribution of the ^3He particles in the centre-of-mass co-ordinate system.

2. – Experimental method.

In general the method closely followed that used in experiments on the photoprotons emitted from oxygen, carbon and nitrogen (8). The narrow bremsstrahlung beam entered a high-pressure helium chamber through a lead collimator 35 cm thick and the charged particles emitted from the gas were detected in photoplates disposed symmetrically about the beam. In preliminary experiments it was found that most of the particles recorded were

(4) E. G. FULLER: *Phys. Rev.*, **96**, 1306 (1954).
(5) J. M. REID, P. SWINBANK and J. R. ATKINSON: *Physica*, **22**, 1142A (1956). J. M. REID: private communication (1958).
(6) G. DE SAUSSURE and L. S. OSBORNE: *Phys. Rev.*, **99**, 843 (1955).
(7) A. N. GORBUNOV and V. M. SPIRIDONOV: *Žu. Éksper. Teor. Fiz.*, **33**, 21 (1957).
(8) D. L. LIVESEY: *Can. Journ. Phys.*, **34**, 1022 (1956); **35**, 987 (1957).

protons produced in one or other of the reactions

$$^4\mathrm{He} + h\nu = {}^1\mathrm{H} + {}^3\mathrm{H} - 19.8\ \mathrm{MeV},$$

$$^4\mathrm{He} + h\nu = {}^1\mathrm{H} + {}^1\mathrm{n} + {}^2\mathrm{H} - 26.0\ \mathrm{MeV},$$

$$^4\mathrm{He} + h\nu = 2{}^1\mathrm{H} + 2{}^1\mathrm{n} - 28.2\ \mathrm{MeV}.$$

There were however several dense tracks possibly due to ^{3}He produced in the (γ, n) reaction

$$^4\mathrm{He} + h\nu = {}^1\mathrm{n} + {}^3\mathrm{He} - 20.6\ \mathrm{MeV}.$$

One of the plates used (hereafter called Plate A) was of Ilford type E-1, and grain-counts were carried out on all tracks longer than 30 μm in a limited area of this plate. Most of the tracks were shown to be protons by comparison with proton tracks produced within the emulsion by the fast neutron background. A second group consisted of short tracks as dense as those of radioactive α-particles, and there was also an intermediate group ascribed to deuterons and tritons. Many of the supposed ^{3}He tracks were too short and too dense to be suitable for grain-counting so a different method had to be used for their identification.

In subsequent experiments the photoplates were reversed in their holders immediately after exposure and a weak source of Actinium A, B and C was placed at the centre of the chamber which was immediately evacuated. This further exposure (of 30 minutes) introduced α-particles of ranges up to 30 μm into the plates from the « wrong » direction. The α-particle tracks occurred in most microscope fields and greatly facilitated identification of the ^{3}He tracks. Since the ionization density in a short ^{3}He track is very nearly equal to that produced by ^{4}He, a criterion was adopted whereby a supposed ^{3}He track was accepted only if it was at least as dense as a nearby α-particle track. As a result a few genuine ^{3}He tracks may have been rejected, but elimination of triton tracks, numerous in these plates, should have been ensured.

A check on the reliability of the method was provided by variations in the processing of the plates. In the series of Ilford C-2 plates (B, C, D and E) the development was successively reduced, so that while plate B received overnight immersion in cold HQ developer ([9]) plus 30 minutes in HQ at 20 °C, plate E received the cold immersion only. These plates showed considerable variation in the density of proton tracks, but investigation of similar areas revealed that the proportion of ^{3}He tracks accepted was $(12 \pm 1)\%$ of all

([9]) W. K. Dawson and D. L. Livesey: *Can. Journ. Phys.*, **34**, 241 (1956).

tracks exceeding 10 μm in length within the limits of experimental error. This ratio did not depend on the local abundance of Actinium α-particles or on the area selected.

A possible source of error lay in the occurrence of recoiling ^{4}He nuclei ejected from the gas by fast neutrons. This was checked experimentally by measurements of the fast neutron flux, both within the beam and in the region of the plates, and by counting the number of helium tracks entering plate A in the « wrong » direction. The results showed that less than 5% of the recorded tracks could be due to this process. Radioactive contamination could also cause background effects, but no definite α-particles could be detected in an exposure without gas in the chamber. A survey of possible sources of error indicated a maximum systematic error of 10%, but this is in the opposite sense to the errors caused by rejecting too many ^{3}He tracks.

In the final examination of the plates the two observers worked independently, scanning considerable areas at the edges nearest the beam, and also at the far edges to check the expected variation of yield with distance from the beam. The photon energy and angle of emission in the center-of-mass co-ordinates were calculated for each ^{3}He track accepted. The ^{3}He energy varies rapidly with laboratory angle for a given photon energy; the lowest ^{3}He energy recorded (4.5 MeV) corresponded to a photon of 34 MeV at the smallest useful angle of 30° and to a photon of 45 MeV at the maximum angle of 150°. Accordingly a correction was applied for the loss of tracks at backward angles for photon energies below 45 MeV. An additional correction for the events occurring outside the useful range of angles amounted to not more than 5% above 45 MeV.

The integrated (γ, n) cross-sections for various ranges of photon energy were calculated from the expression:

$$n = (F\pi r^2)\, N(y/R^2)(\Sigma/2\pi)\,,$$

where n = number of events recorded in a certain energy range per unit area of plate,

F = effective photon flux per unit area of the beam and per unit energy range,

r = radius of the collimated beam,

N = number of helium nuclei per unit volume of gas,

y = perpendicular distance of the beam axis from the plane of the photoplate,

R = mean distance from the beam axis to the scanned area

and Σ = integrated cross-section over the chosen energy range.

Corrections were worked out for the effects of beam polarization, finite beam cross-section and finite scanned area, but in no case did these exceed 2%. The value of F was calculated from the X-ray dosage in nominal roentgens indicated by the synchrotron monitor, which was calibrated in a separate experiment on the activation of copper foils by the reaction $^{63}Cu(\gamma, n)^{62}Cu$. The integrated cross-section for this reaction was taken to be 0.60 MeV barns from the threshold up to 70 MeV ([10]). This figure led to a direct estimate of the photon flux at 18 MeV in the bremsstrahlung spectrum. At all other photon energies the flux F was calculated by normalizing a Schiff curve to the point fixed at 18 MeV. It should be pointed out that the maximum synchrotron energy was not stabilised and the procedure may not be reliable above 60 MeV; at lower energies the values of F should be accurate to $\pm 10\%$.

3. – Results and discussion.

The detailed results obtained for the integrated cross-sections in different energy regions are shown in Table I, where the standard errors are based on statistical fluctuations only. The different plates and observers gave quite consistent results and the final mean values were weighted according to the areas of plate scanned. The total integrated cross-section from 40 MeV to 60 MeV amounts to 7.5 MeV mb with a standard error of $\pm$ 0.9 MeV mb

TABLE I.

Plate	Observer	Area cm²	Mean R cm	Integrated cross-sections (MeV mb)		
				(40÷45) MeV(*)	(45÷50) MeV	(50÷60) MeV
A	DLL	1.70	3.5	2.4 ±0.5	1.9 ±0.5	2.3±0.6
B	DLL	1.50	3.5	2.6 ±0.5	3.1 ±0.6	2.3±0.6
B	DLL	1.50	4.4	3.4 ±0.8	2.85±0.75	2.3±0.8
C	IGM	4.00	3.5	2.2 ±0.5	1.75±0.4	2.4±0.5
D	DLL	2.00	3.3	2.9 ±0.6	3.2 ±0.7	3.2±0.8
E	DLL	4.00	3.55	2.8 ±0.45	2.25±0.45	2.8±0.6
E	IGM	2.80	4.4	2.2 ±0.7	1.5 ±0.5	3.2±0.9
			Weighted means:	2.57±0.25	2.21±0.25	2.7±0.3
			Total cross-section:	7.5±0.5 MeV mb		

(*) Corrected for losses at angles exceeding 110°.

([10]) A. I. BERMAN and K. L. BROWN: *Phys. Rev.*, **96**, 83 (1954).

after full allowance has been made for all errors arising in the experiment. In comparison, NICOLAI and GOLDWASSER found a value of 28 MeV mb for the entire energy range up to 135 MeV, and GORBUNOV and SPIRIDONOV obtained about 6.8 MeV mb for the (γ, p) reaction between 40 MeV and 60 MeV.

The shape of the cross-section curve was found by plotting the data at 2.5 MeV energy intervals, as shown in Fig. 1. The results were everywhere consistent with a cross-section falling smoothly from an estimate of (0.63 ± 0.10) mb at 40 MeV to (0.25 ± 0.04) mb at 60 MeV, with a best value of (0.37 ± 0.04) mb at 50 MeV. This curve agrees closely with the (γ, p) data of GORBUNOV and SPIRIDONOV but falls rather more rapidly than the earlier (γ, n) curve of DE SAUSSURE and OSBORNE. The new results should reduce slightly the bremsstrahlung-weighted cross-section as estimated by RUSTGI and LEVINGER [11], but their theoretical conclusions and those of FOLDY [12] are not materially affected.

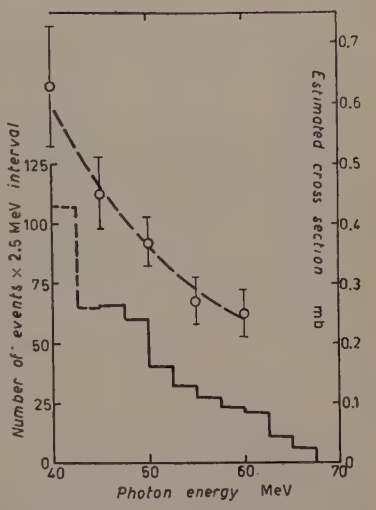

Fig. 1. – Lower: number of (γ, n) events plotted against photon energy. Upper: estimated (γ, n) cross-section as a function of photon energy; the smooth curve has been fitted to give the best agreement with the integrated cross-sections shown in Table I.

In order to obtain the final angular distribution of Fig. 2, the distribution of ^3He tracks in centre-of-mass co-ordinates was corrected for the variation of the effective solid angle of collection with angle in the laboratory system. The difference between the centre-of-mass angle and laboratory angle is very nearly constant over a wide energy range, so the correction was applied by dividing each ordinate by the sine of the corresponding mean laboratory angle. The distribution for all events above 45 MeV shows a distinct forward peak near 70° and the data above 50 MeV are also asymmetrical about 90°. Attempts were made to represent the distributions by functions of the form $f(\theta) = A + B \sin^2\theta\,(1 + C \cos\theta)$. In all cases the value of A/B was not significantly different from zero. The best fit was obtained with C approximately equal to 0.5, but experimental errors at the extreme angles precluded a real test of the function $f(\theta)$. The earlier work of DESAUSSURE and OSBORNE gave al-

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[11] M. L. RUSTGI and J. S. LEVINGER: *Phys. Rev.*, **106**, 530 (1957).
[12] L. L. FOLDY: *Phys. Rev.*, **107**, 1303 (1957).

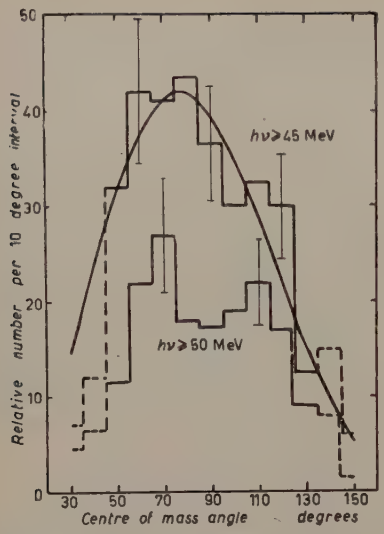

Fig. 2. – Angular distribution of ^{3}He particles emitted in the C.M. co-ordinates, after correction for the variation of solid angle of collection with angle in the laboratory system. Lower: all events above 50 MeV. Upper: all events above 45 MeV, the curve representing the function $f(\theta) = B \sin^2\theta\,(1+0.5\cos\theta)$.

most a pure $\sin^2\theta$ distribution for the energy range 36 MeV to 84 MeV. The results of GORBUNOV and SPIRIDONOV on the (γ, p) reaction are also in agreement with a very low value A/B; their value of C is 1.05 ± 0.16 for all events between 30 MeV and 170 MeV.

A test of the effects of the cosine term is provided by calculations of the ratio of events at forward angles to those at backward angles. In our work this is 1.4 ± 0.2 for energies above 45 MeV, whereas GORBUNOV and SPIRIDONOV obtained 2.4 for photoprotons in the same energy region. The two cases are not strictly comparable because the (γ, n) reaction involves the emission of only one charged particle. The forward predominance in ^{3}He emission implies backward emission of neutrons, such as occurs in the photodisintegration of the deuteron at high energies. Apart from differences in the values of C, the (γ, n) and (γ, p) reactions appear to have similar properties in the energy region investigated.

* * *

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RIASSUNTO (*)

Il metodo delle lastre fotografiche è stato usato per rivelare particelle di ^{3}He emesse da elio gassoso esposto a bremsstrahlung di energia massima 70 MeV. La sezione d'urto ^{4}He(γ, n)^{3}He si trova variare tra valori di $(0{,}63 \pm 0{,}10)$ mb a 40 MeV e $(0{,}25 \pm 0{,}04)$ mb a 60 MeV, la sezione d'urto integrata essendo 7.5 MeV mb tra questi limiti. La distribuzione angolare delle particelle di ^{3}He nel sistema del centro di massa ha un picco in direzione anteriore e si può descrivere con l'espressione $f(\theta) = B \sin^2\theta\,(1+C\cos\theta)$ con C approssimativamente uguale a 0.5.

(*) *Traduzione a cura della Redazione.*

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On the Bound State Problem in Quantum Field Theory (*).

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Summary. — A causal and invariant scalar field involving a stable bound state is investigated. A formula for the S-matrix is derived and it is shown that the bound state can be described by a local and invariant field operator. For simplicity only the case of spin zero particles and bound states is considered; however, the extension to other cases is possible.

Introduction.

For a relativistic quantum mechanical system with an energy momentum operator P_μ the one particle states are defined as eigen states of $-P_\mu^2$ with a discrete non-vanishing rest mass ([1,2]). In general, there may be several kinds of particles, each characterized by a certain value of rest mass, spin, charge, etc. Usually a division is made between elementary and composite particles. But it seems to be hard to define this distinction in a convincing manner. In the conventional formulation of quantum field theory each of the elementary

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([1]) For a discussion of the particle aspect in quantum field theory compare R. HAAG: *Dan. Mat. Fys. Medd.*, **29**, 12 (1955), especially chap. I, no. 1.

([2]) We exclude the case of particles with zero rest mass.

particles is described by a basic field operator whereas the composite particles (like the deuteron in the ground state, etc.) appear as the stable bound states of the system. But this definition of a composite particle depends, of course, on the formalism. The same particle which is regarded as an elementary particle in one formalism may be a composite particle in another ([3,4]).

In this paper we want to investigate whether the principle of microscopic causality sheds any new light on this question. We consider a model of a causal and invariant scalar field $A(x)$ describing just two kinds of particles, an elementary particle of mass m and a composite particle of mass M, both of spin zero. It will be shown that it is possible to define a field operator $B(x)$ for the composite particle (explicitly expressed in terms of the original field $A(x)$) which is in some respect analogous to the field operator of an elementary particle. The invariant operator $B(x)$ satisfies the requirement of microscopic causality, furthermore the S-matrix can be expressed by the vacuum τ-functions ([5]) of the field operator $A(x)$ and $B(x)$ exactly in the same way as in the case of elementary particles only. Therefore the principle of microscopic causality offers no possibility of distinguishing between elementary and composite particles. With respect to the S-matrix the elementary particles as well as the composite particles of the model are described by invariant field operators which satisfy the requirement of microscopic causality ([6,7]).

The formalism developed in this paper is closely related to other recent investigations. Starting from similar requirements one can derive the same results by applying the method of strong operator convergence in the form developed by HAAG ([8,9]). An equivalent formalism was also obtained from the recursion formulae for retarded functions recently proposed by NISHIJIMA as an axiomatic formulation of quantum field theory ([10,11]).

([3]) Compare E. FERMI and C. N. YANG: *Phys. Rev.*, **76**, 1739 (1949). In this paper the hypothesis that π-mesons may be composite particles is discussed.

([4]) In this connection an interesting example is Gürsey's model of a theory of elementary particles (F. GÜRSEY: to be published). There, following a suggestion of Heisenberg, all particles appear as composite and no elementary particles correspond to the basic fields.

([5]) The vacuum τ-functions are defined as the vacuum expectation values of multiple, time ordered, operator products.

([6]) This possibility was first mentioned by N. BOGOLJUBOV: unpublished lecture notes.

([7]) The S-matrix of the model considered is causal according to the definition given in H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **6**, 319 (1957).

([8]) R. HAAG: *Proc. of the Lille Conference on Mathematical problems of the quantum theory of fields* (1957), in print.

([9]) R. HAAG: preprint, to be published.

([10]) K. NISHIJIMA: *Progr. Theor. Phys.*, **17**, 765 (1957).

([11]) K. NISHIJIMA: preprint, to be published.

1. – General conditions.

We consider the model of a neutral scalar field described by a hermitian operator $A(x)$ assumed to be invariant under the inhomogeneous Lorentz group. We assume that the principle of microscopic causality

$$[A(x), A(y)] = 0 \qquad \text{for } (x-y)^2 > 0 \tag{1}$$

holds and that no negative eigenvalues appear in the energy and rest mass spectrum. The operators $A(x)$ will be supposed to form an irreducible operator ring. For simplicity we assume that there are just two non-vanishing discrete eigenvalues m^2 and M^2 of $-P_\mu^2$ and that

$$(\Omega, A(x)\Phi) \not\equiv 0 \qquad \text{if } -P_\mu^2\Phi = m^2\Phi$$

but

$$\left.\begin{aligned} (\Omega, A(x)\Psi) &\equiv 0 \\ (\Omega, A(x)A(y)\Psi) &\not\equiv 0 \end{aligned}\right\} \quad \text{if } -P_\mu^2\Psi = M^2\Psi .$$

In addition we suppose that the states Φ and Ψ have spin zero.

This situation may occur for example in the case of A^4-coupling if there is an elementary particle of mass m and a stable two particle bound state of the mass M.

In order to describe the bound states we introduce the operator [12]

$$B(x, \xi) = TA(x+\xi)A(x-\xi) \tag{2}$$

and define [13] incoming fields, according to WIGHTMAN [14], by

$$\left\{\begin{aligned} A_{\text{in}}(x) &= A(x) + \int \Delta_{\text{Ret}}(m, x-x')j(x')\,\mathrm{d}x', \\ B_{\text{in}}(x, \xi) &= B(x, \xi) + \int \Delta_{\text{Ret}}(M, x-x')j(x', \xi)\,\mathrm{d}x', \end{aligned}\right. \tag{3}$$

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[12] ξ may be taken as spacelike or timelike 4-vector, but it is supposed that $\xi^2 \neq 0$.

[13] The integral expressions in eq. (3) are to be understood in the sense of weak operator convergence which means that

$$(\Phi, A_{\text{in}}(x)\Psi) = (\Phi, A(x)\Psi) + \int \Delta_{\text{Ret}}(m, x-x')(\Phi, j(x')\Psi)\,\mathrm{d}x',$$

between any normalizable state vectors Φ and Ψ.

[14] The following definition of the asymptotic fields $A_{\text{in}}(x)$, $A_{\text{out}}(x)$ was first suggested by WIGHTMAN: private communication.

with the current operators

$$(3') \qquad \begin{cases} j(x) = (\Box_x - m^2)A(x)\,, \\ j(x,\xi) = (\Box_x - M^2)B(x,\xi)\,, \end{cases} \qquad \Box_x = \sum \frac{\partial^2}{\partial x_\mu^2}\,.$$

The outgoing fields $A_{\text{out}}(x)$, $B_{\text{out}}(x,\xi)$ are correspondingly defined with the help of the advanced functions $\Delta_{\text{Adv}}(m,x)$ and $\Delta_{\text{Adv}}(M,x)$. The general conditions under which the so defined operators $A_{\text{in}}^{\text{out}}(x)$ exist have been given and investigated in details by GREENBERG and can easily be extended to the case of the operators $B_{\text{in}}^{\text{out}}(x,\xi)$ ([15,16]).

Finally we demand that the incoming field operators $A_{\text{in}}(x)$, $B_{\text{in}}(x,\xi)$ together form an irreducible operator ring and correspondingly the outgoing fields $A_{\text{out}}(x)$, $B_{\text{out}}(x,\xi)$.

So far we have listed the general requirements which we need in the work which follows. We conclude this section by deriving some simple properties of the operators $A_{\text{in}}^{\text{out}}(x)$, $B_{\text{in}}^{\text{out}}(x,\xi)$. As a consequence of definition (3) they are solutions of the Klein-Gordon equations for the masses m and M, respectively:

$$(4) \qquad \begin{cases} (\Box_x - m^2)A_{\text{in}}^{\text{out}}(x) = 0\,, \\ (\Box_x - M^2)B_{\text{in}}^{\text{out}}(x,\xi) = 0\,. \end{cases}$$

We have the invariance properties

$$(5) \qquad \frac{\partial A_{\text{in}}^{\text{out}}(x)}{\partial x_\mu} = -i[P_\mu, A_{\text{in}}^{\text{out}}(x)]\,, \qquad \frac{\partial B_{\text{in}}^{\text{out}}(x,\xi)}{\partial x_\mu} = -i[P_\mu, B_{\text{in}}^{\text{out}}(x,\xi)]\,,$$

$$A_{\text{in}}^{\text{out}}(Lx) = U(L)A_{\text{in}}^{\text{out}}(x)U(L)^{-1}, \qquad B_{\text{in}}^{\text{out}}(Lx, L\xi) = U(L)B_{\text{in}}^{\text{out}}(x,\xi)U(L)^{-1},$$

for an arbitrary Lorentz transformation L. ($U(L)$ denotes the unitary operator transforming $A(x)$ into $A(Lx)$.) These invariance properties are easy to prove if (3) is written in momentum space.

As a consequence of (4), (5) the vacuum expectation values of the incoming fields vanish:

$$(6) \qquad (\Omega, A_{\text{in}}^{\text{out}}(x)\Omega) = 0\,, \qquad (\Omega, B_{\text{in}}^{\text{out}}(x,\xi)\Omega) = 0\,.$$

For example,

$$M^2(\Omega, B_{\text{in}}(x,\xi)\Omega) = \sum \frac{\partial^2}{\partial x_\mu^2}(\Omega, B_{\text{in}}(x,\xi)\Omega) = 0\,,$$

because $P_\mu\Omega = 0$.

([15]) O. W. GREENBERG and A. S. WIGHTMAN: preprint, to be published.

([16]) In Sect. 2 we will use somewhat more restrictive conditions than Greenberg, for details see reference ([20]).

Furthermore $A_{\text{in}}^{\text{out}}(x)$ and $B_{\text{in}}^{\text{out}}(x, \xi)$ satisfy the asymptotic conditions

$$\left\{\begin{array}{l} \lim\limits_{t\to\pm\infty} (\Phi, A_f(t)\Psi) = (\Phi, A_{\text{in}}^{\text{out}} {}_f \Psi), \\ \lim\limits_{t\to\pm\infty} (\Phi, B_F(t, \xi)\Psi) = (\Phi, B_{\text{in}}^{\text{out}} {}_F(\xi)\Psi), \end{array}\right. \tag{7}$$

with

$$\begin{aligned} B_F(t, \xi) &= -i\int_{x_0=t} \mathrm{d}_3 x \left\{B(x, \xi)\frac{\partial}{\partial x_0} F^*(x) - F^*(x)\frac{\partial}{\partial x_0} B(x, \xi)\right\}, \\ B_{\text{in}}^{\text{out}} {}_F(\xi) &= -i\int \mathrm{d}_3 x \left\{B_{\text{in}}^{\text{out}}(x, \xi)\frac{\partial}{\partial x_0} F^*(x) - F^*(x)\frac{\partial}{\partial x_0} B_{\text{in}}^{\text{out}}(x, \xi)\right\}, \end{aligned} \tag{7'}$$

(correspondingly the definition of A_f and $A_{\text{in}}^{\text{out}} {}_f$) for any normalizable solution $f(x)$, $F(x)$ of the Klein-Gordon equation

$$(\Box - m^2) f(x) = 0,$$

$$(\Box - M^2) F(x) = 0.$$

Eq. (7) can be proved by forming the integral expression (7') for both sides of (3) and taking the limit $t \to \pm\infty$.

2. – Commutation relations for the asymptotic fields.

2·1. *Elementary particles.* – Our first aim is to derive commutation relations for the incoming (or outgoing) fields defined in Sect. 1. We begin with the case of elementary particles and prove the relation [17]

$$[A_{\text{in}}(x),\ A_{\text{in}}(y)] = [A_{\text{out}}(x), A_{\text{out}}(y)] = i\,\Delta(m, x-y) \tag{8}$$

if $A(x)$ is normalized in the following manner. Let Φ_k be a one-particle state with energy momentum eigenvalue k_μ

$$\left\{\begin{array}{l} P_\mu \Phi_k = k_\mu \Phi_k, \\ (\Phi_k, \Phi_{k'}) = 2k_0^m \delta_3(\boldsymbol{k} - \boldsymbol{k}'), \end{array}\right. \qquad -k_\mu^2 = m^2, \quad k_0^m = |\sqrt{\boldsymbol{k}^2 + m^2}|.$$

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[17] This was already shown by GREENBERG, reference [15] under the assumption that the equal time commutator $[A(x)\dot{A}(y)]_{x_0=y_0}$ is a c-number. Here we use a different method which can be extended to the case of bound states.

Then it follows from translation invariance that

$$(\Omega, A(x)\Phi_k) = c\,\frac{\exp[ikx]}{(2\pi)^{\frac{3}{2}}},$$

with the constant

$$c = (2\pi)^{\frac{3}{2}}(\Omega, A(0)\Phi_k).$$

Now $A(x)$ shall be normalized by the condition $c = 1$.

It is easy to determine the matrix elements of $A_{\text{in}}(x)$ between the vacuum state and an arbitrary state vector. From the definition (3) it follows that

$$(\Omega, A_{\text{in}}(x)\Phi_k) = (\Omega, A(x)\Phi_k) = \frac{\exp[ikx]}{(2\pi)^{\frac{3}{2}}},$$

for the one-particle state Φ_k. On the other hand if Φ is an eigenstate of $-P_\mu^2$ with a rest mass $\varkappa^2 \neq m^2$ we have

$$(\Omega, A_{\text{in}}(x)\Phi) = 0$$

because

$$(m^2 - \varkappa^2)(\Omega, A_{\text{in}}(x)\Phi) = 0$$

follows from (4) and (5).

With this result we can calculate the vacuum expectation value of the commutator (8). We see that

$$(\Omega, A_{\text{in}}(x)A_{\text{in}}(y)\Omega) = \int \frac{\mathrm{d}_3 k}{2k_0^m}(\Omega, A_{\text{in}}(x)\Phi_k)(\Phi_k, A_{\text{in}}(y)\Omega) = i\,\Delta^+(m, x-y).$$

Hence,

$$(\Omega, [A_{\text{in}}^{\text{out}}(x)A_{\text{in}}^{\text{out}}(y)]\,\Omega) = i\,\Delta(m, x-y). \tag{9}$$

Now we turn to the operator form (8) and using the conditions of Sect. **1** we shall show that

(i) the commutators of the incoming and the outgoing fields coincide:

$$[A_{\text{out}}(x), A_{\text{out}}(y)] = [A_{\text{in}}(x), A_{\text{in}}(y)] \tag{10}$$

and that

(ii) the commutators of the incoming field is a c-number:

$$[A_{\text{in}}(x), A_{\text{in}}(y)] = (\Omega, [A_{\text{in}}(x), A_{\text{in}}(y)]\Omega). \tag{11}$$

Relation (8) is, of course, a consequence of the statements (i), (ii) and Eq. (9).

In order to prove (i) and (ii) it is convenient to expand the field operators with respect to a complete orthonormal system $\{f_\alpha(x)\}$ of the positive frequency solutions of $(\Box - m^2) f(x) = 0$:

$$A(x) = \sum f_\alpha(x) A_\alpha^+(x_0) + \sum f_\alpha^*(x) A_\mu^-(x_0)\,,$$
$$A_{\text{in}}^{\text{out}}(x) = \sum f_\alpha(x) A_{\text{in}}^{+\,\text{out}}{}_\alpha + \sum f_\alpha^*(x) A_{\text{in}}^{-\,\text{out}}{}_\alpha\,.$$

For the proof of statement (i) we start with the identity [18]

$$(12)\quad \int \mathrm{d}x \int \mathrm{d}y f_\alpha^*(x) f_\beta(y) K_x^m K_y^m T A(x) A(y) = \int \mathrm{d}y \int \mathrm{d}x\, f_\alpha^*(x) f_\beta(y) K_x^m K_y^m T A(x) A(y)\,,$$

$$K_x^m = \sum \frac{\partial^2}{\partial x_\mu^2} - m^2\,.$$

This relation is not self evident because there are simple examples of pathological fields which do not satisfy eq. (12). But using the fact that as a consequence of causality and spectrum conditions the vacuum expectation values

$$\left(\Omega, A(x_1) \cdots A(x_n) T(A(x), A(y)) A(y_1) \cdots A(y_m) \Omega\right)$$

are boundary values of Wightman's analytical functions [19] a more detailed investigation [20] justifies the interchanging of the x- and y-integration in

$$\int \mathrm{d}x \int \mathrm{d}y\, f_\alpha^*(x) f_\beta(y) K_x^m K_y^m (\Omega, A(x_1) \ldots A(x_n) T(A(x), A(y)) A(y_1) \ldots A(y_m) \Omega)\,.$$

Then relation (12) holds for every matrix element of $TA(x)A(y)$ because, according to the irreducibility of the operators $A(x)$, any state vector can be written as a linear superposition of vectors $A(x_n) \ldots A(x_1)\Omega$.

Rearranging the integral on the left hand side with the help of Green's theorem we get

$$-i \int \mathrm{d}y f_\beta(y) K_y^m T A(x) A(y) = -i \int \mathrm{d}y_0 \frac{\partial}{\partial y_0} \int \mathrm{d}_3 y\, T A(x) A(y) \overleftrightarrow{\frac{\partial}{\partial y_0}} f_\beta(y) =$$
$$= A(x) A_{\text{in}\beta}^- - A_{\text{out}\beta}^- A(x)\,,$$

$$\left(f(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} g(x) = f(x) \frac{\partial g(x)}{\partial x_0} - g(x) \frac{\partial f(x)}{\partial x_0}\right).$$

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[18] In the sense of weak operator convergence (compare footnote [13]), correspondingly for all following operator expressions containing time integration.

[19] A. S. WIGHTMAN: *Phys. Rev.*, **101**, 860 (1956).

[20] W. ZIMMERMANN: *Order of integrations in reduction formulae*, unpublished manuscript.

Carrying out the integration over x in the same way, we obtain

$$\int \mathrm{d}x \int \mathrm{d}y f^*_\nu(x) f_\beta(y) K^m_x K^m_y T A(x) A(y) = A^-_{\mathrm{out}\beta} A^+_{\mathrm{out}\nu} - A^-_{\mathrm{out}\beta} A^+_{\mathrm{in}\nu} - A^+_{\mathrm{out}\nu} A^-_{\mathrm{in}\beta} + A^+_{\mathrm{in}\nu} A^-_{\mathrm{in}\beta} .$$

For the integral on the right hand side of (12) we have, correspondingly

$$\int \mathrm{d}y \int \mathrm{d}x f^*_\nu(x) f_\beta(y) K^m_x K^m_y T A(x) A(y) = A^+_{\mathrm{out}\nu} A^-_{\mathrm{out}\beta} - A^+_{\mathrm{out}\nu} A^-_{\mathrm{in}\beta} - A^-_{\mathrm{out}\beta} A^+_{\mathrm{in}\nu} + A^-_{\mathrm{in}\beta} A^+_{\mathrm{in}\nu} .$$

Inserting these expressions into (12), we get

$$[A^+_{\mathrm{out}\alpha} A^-_{\mathrm{out}\beta}] - [A^+_{\mathrm{in}\alpha} A^-_{\mathrm{in}\beta}] = 0 .$$

With the corresponding relations between $A^+_{\mathrm{in}\nu}$ and $A^+_{\mathrm{in}\beta}$ or $A^-_{\mathrm{in}\nu}$ and $A^-_{\mathrm{in}\beta}$ statement (i) follows:

$$[A_{\mathrm{in}}(x), A_{\mathrm{in}}(y)] = [A_{\mathrm{out}}(x), A_{\mathrm{out}}(y)] .$$

The second statement (ii) may be derived from the identity

$$(13) \quad \int \mathrm{d}x \int \mathrm{d}y f^*_\alpha(x) f_\beta(y) K^m_x K^m_y T A(x) A(y) A(z) = \\ = \int \mathrm{d}y \int \mathrm{d}x f^*_\alpha(x) f_\beta(y) K^m_x K^m_y T A(x) A(y) A(z) ,$$

which can again be rearranged with the help of Green's theorem. The final result is

$$A(z)[A^+_{\mathrm{in}\alpha} A^-_{\mathrm{in}\beta}] = [A^+_{\mathrm{out}\alpha} A^-_{\mathrm{out}\beta}] A(z) .$$

Using statement (i), we get

$$[[A^+_{\mathrm{in}\alpha} A^-_{\mathrm{in}\beta}] A(z)] = 0 ,$$

and this shows that $[A^+_{\mathrm{in}\alpha}, A_{\mathrm{in}\beta}]$ is a c-number because we have assumed that the $A(x)$ form an irreducible operator ring. With the corresponding results for $[A^+_{\mathrm{in}\alpha}, A^+_{\mathrm{in}\beta}]$, $[A^-_{\mathrm{in}\alpha}, A^-_{\mathrm{in}\beta}]$, we have proved the statement (ii).

2·2. *Bound states.* – In the next step we want to derive commutation relations for the field operators $B^{\mathrm{out}}_{\mathrm{in}}(x, \xi)$. We begin with the proof of the following statements:

(i) The commutators of the incoming and outgoing fields coincide:

$$[B_{\text{out}}(x,\xi)\,B_{\text{out}}(y,\eta)] = [B_{\text{in}}(x,\xi)\,B_{\text{in}}(y,\eta]\,. \tag{14}$$

(ii) The commutator of the incoming field is a c-number:

$$[B_{\text{in}}(x,\xi)B_{\text{in}}(y,\eta)] = (\Omega, [B_{\text{in}}(x,\xi)B_{\text{in}}(x,\eta)]\Omega)\,. \tag{15}$$

We expand the operators B and $B^{\text{out}}_{\text{in}}$ with respect to a complete orthonormal system $\{F_\varkappa(x)\}$ of the positive frequency solutions of $(\Box - M^2)F(x) = 0$:

$$B(x,\xi) \quad = \sum F_\alpha(x)B^+_\alpha(x_0,\xi) + \sum F^*_\alpha(x)B^-_\alpha(x_0,\xi)\,,$$

$$B^{\text{out}}_{\text{in}}(x,\xi) = \sum F_\alpha(x)B^{+}_{\text{in}\,\alpha}(\xi) + \sum F^*_\alpha(x)B^{-}_{\text{in}\,\alpha}(\xi)\,.$$

The coefficients are determined by [21]

$$B^\pm_\varkappa(x_0,\xi) = \mp i\int \mathrm{d}_3x\left\{B(x,\xi)\,\frac{\partial}{\partial x_0}\,F^\mp_\alpha(x) - F^\mp_\alpha(x)\,\frac{\partial}{\partial x_0}\,B(x,\xi)\right\},$$

$$F^+_\alpha(x) = F_\alpha(x),\ F^-_\alpha(x) = F^*_\alpha(x),$$

(and similarly for $B^{\pm}_{\text{in}\,\varkappa}(\xi)$).

As a consequence of causality and spectrum conditions we have the identity

$$\int \mathrm{d}x \int \mathrm{d}y\, F^*_\varkappa(x)F_\beta(y)K^M_x K^M_y T A(x+\xi)A(x-\xi)A(y+\eta)A(y-\eta) = \tag{16}$$

$$= \int \mathrm{d}y \int \mathrm{d}x\, F^*_\varkappa(x)F_\beta(y)K^M_x K^M_y T A(x+\xi)A(x-\xi)A(y+\eta)A(y-\eta)\,.$$

After integration over x_0 and y_0, this relation yields

$$[B^+_{\text{out}\,\varkappa}(\xi),\ B^-_{\text{out}\beta}(\eta)] - [B^+_{\text{in}\alpha}(\xi),\ B^-_{\text{in}\beta}(\eta)] = 0\,,$$

and similarly for $[B^\pm_{\text{out}\,\alpha}(\xi),\ B^\pm_{\text{out}\beta}(\eta)]$. So we have proved that

$$[B_{\text{out}}(x,\xi),\ B_{\text{out}}(y,\eta)] = [B_{\text{in}}(x,\xi),\ B_{\text{in}}(y,\eta)]\,.$$

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[21] If ξ is spacelike we have

$$B^-_\alpha(x_0,\xi) = B^+_\alpha(x_0,\xi)^*\,.$$

This relation is no longer true if ξ is a timelike vector.

Statement (ii) may be derived from

$$(17)\quad \int \mathrm{d}x \int \mathrm{d}y\, F^*_\alpha(x) F_\beta(y) K_x^M K_y^M T A(x+\xi) A(x-\xi) A(y+\eta) A(y-\eta) A(z) =$$
$$= \int \mathrm{d}y \int \mathrm{d}x\, F^*_\alpha(x) F_\beta(y) K_x^M K_y^M T A(x+\xi) A(x-\xi) A(y+\eta) A(y-\eta) A(z)\,,$$

which yields

$$[B^+_{\mathrm{out}\alpha}(\xi)\, B^-_{\mathrm{out}\beta}(\eta)]\, A(z) = A(z)[B^+_{\mathrm{in}\alpha}(\xi)\, B^-_{\mathrm{in}\beta}(\eta)]$$

or

$$[[B^+_{\mathrm{in}\alpha}(\xi)\, B^-_{\mathrm{in}\beta}(\eta)]\, A(z)] = 0\,.$$

Therefore the commutator $[B_{\mathrm{in}}(x,\xi)\, B_{\mathrm{in}}(y,\eta)]$ is a c-number (statement ii).

Now we can determine the commutator (15) by calculating the vacuum expectation value. For this purpose we determine first the matrix elements of $B_{\mathrm{in}}(x,\xi)$ between the vacuum state and an arbitrary state vector.

If Φ_k is an eigenstate of P_μ belonging to the eigenvalue k_μ and the rest mass $-k_\mu^2 = M^2$ we have

$$(\Omega,\, TA(x+\xi) A(x-\xi) \Phi_k) = \frac{\exp\,[ikx]}{(2\pi)^{\frac{3}{2}}}\, F_k(\xi)\,,$$

with

$$(18)\quad F_k(\xi) = (2\pi)^{\frac{3}{2}} (\Omega,\, TA(\xi) A(-\xi) \Phi_k)$$

and

$$(\Box_x + M^2)(\Omega,\, TA(x+\xi) A(x-\xi) \Phi_k) = 0\,.$$

From this and the definition (3) of $B_{\mathrm{in}}(x,\xi)$ it follows that

$$(19)\quad (\Omega,\, B_{\mathrm{in}}(x,\xi)\Phi_k) = (\Omega,\, B(x,\xi)\Phi_k) = \frac{\exp\,[ikx]}{(2\pi)^{\frac{3}{2}}}\, F_k(\xi)\,.$$

On the other hand if Φ is an eigenstate belonging to the eigenvalue k_μ with the rest mass $-k_\mu^2 \neq M^2$ we get from (5):

$$(k_\mu^2 + M^2)(\Omega,\, B_{\mathrm{in}}(x,\xi)\Phi) = -(\Box_x - M^2)(\Omega,\, B_{\mathrm{in}}(x,\xi)\Phi) = 0$$

hence

$$(\Omega,\, B_{\mathrm{in}}(x,\xi)\Phi) = 0\,.$$

Therefore we have

$$(\Omega, B_{\mathrm{in}}(x,\xi) B_{\mathrm{in}}(y,\eta)\Omega) = \int \frac{\mathrm{d}_3 k}{2k_0^M} (\Omega,\, B_{\mathrm{in}}(x,\xi)\Phi_k)\, (\Phi_k,\, B_{\mathrm{in}}(y,\eta)\Omega) =$$
$$= \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}_3 k}{2k_0^M}\, F_k(\xi) F_k(\eta) \exp\,[ik(x-y)]\,,$$

where we have used the invariance under space time reflection for

$$(20) \qquad (\Phi_k, TA(\eta)A(-\eta)\Omega) = (\Omega, TA(\eta)A(-\eta)\Phi_k) = F_k(\eta) .$$

Introducing the Fourier transform $B_{\text{in}}(k, \xi)$, we have

$$B_{\text{in}}(x, \xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}_4 k \exp [ikx] \delta(k^2 + M^2) B_{\text{in}}(k, \xi) ,$$

and defining creation and annihilation operators

$$B^+_{\text{in}}(\boldsymbol{k}, \xi) = B_{\text{in}}(k, \xi) , \qquad \text{for } k_0 = +|\sqrt{\boldsymbol{k}^2 + M^2}|,$$
$$B^-_{\text{in}}(\boldsymbol{k}, \xi) = B_{\text{in}}(-k, \xi) , \qquad \text{for } k_0 = -|\sqrt{\boldsymbol{k}^2 + M^2}|,$$

we get from (19) and (15) the final commutation relations

$$(21) \qquad [B^+_{\text{in}}(\boldsymbol{k}, \xi) B^-_{\text{in}}(\boldsymbol{p}, \eta)] = F_k(\xi) F_p(\eta) 2k_0^M \delta_3(\boldsymbol{k} - \boldsymbol{p}) ,$$

$$(22) \qquad [B^{\pm}_{\text{in}}(\boldsymbol{k}, \xi) B^{\pm}_{\text{in}}(\boldsymbol{p}, \eta)] = 0 .$$

Finally we only mention that the same methods used in this Section to prove the commutation relations (8) and (21) may be applied in order to derive

$$(23) \qquad [A_{\text{in}}(x), B_{\text{in}}(y, \eta)] = 0 .$$

3. – Derivation of an S-matrix formula.

The operators $B^{\text{out}}_{\text{in}}(x, \xi)$ depend on the relative co-ordinate ξ of the bound state. For the S-matrix we are only interested in the center of mass motion of the bound states and want to carry out the limit $\xi \to 0$. Therefore we define operators $B^{\text{out}}_{\text{in}}(x)$ by

$$(24) \qquad \begin{cases} B^{\text{out}}_{\text{in}}(x) = \lim\limits_{\xi \to 0} \dfrac{B^{\text{out}}_{\text{in}}(x, \xi)}{F_0(\xi)} , \\ F_0(\xi) = (2\pi)^{\frac{3}{2}} (\Omega, TA(\xi)A(-\xi)\Phi_0) , \end{cases}$$

where Φ_0 is the bound state at rest.

In order to prove the existence of the limits $\xi \to 0$ in (24) we divide (21) by $F_k(\xi)$ and get

$$\left[\frac{B^+_{\text{in}}(k, \xi)}{F_k(\xi)}, B^-_{\text{in}}(p, \eta)\right] = F_k(\eta) \delta_3(\boldsymbol{k} - \boldsymbol{p}) .$$

Differentiation with respect to ξ_μ yields

$$\left[\frac{\partial}{\partial\xi_\mu}\frac{B^+_{\text{in}}(k,\xi)}{F_k(\xi)}, B^-_{\text{in}}(p,\eta)\right] = 0\,,$$

and correspondingly from (22), (23)

$$\left[\frac{\partial}{\partial\xi_\mu}\frac{B^+_{\text{in}}(k,\xi)}{F_k(\xi)}, B^+_{\text{in}}(p,\eta)\right] = 0\,,$$

$$\left[\frac{\partial}{\partial\xi_\mu}\frac{B^+_{\text{in}}(k,\xi)}{F_k(\xi)}, A^\pm_{\text{in}}(p)\right] = 0\,.$$

Thus $(\partial/\partial\xi_\mu)(B^+_{\text{in}}(k,\xi)/(F_k(\xi))$ commutes with all operators $A_{\text{in}}(x)$ and $B_{\text{in}}(y,\eta)$. Since we have assumed that the operators $A_{\text{in}}(x)$, $B_{\text{in}}(y,\eta)$ together form an irreducible operator ring $(\partial/\partial\xi_\mu)(B^+_{\text{in}}(k,\xi)/F_k(\xi))$ is a c-number:

$$\frac{\partial}{\partial\xi_\mu}\frac{B^+_{\text{in}}(k,\xi)}{F_k(\xi)} = \frac{\partial}{\partial\xi_\mu}\frac{(\Omega, B^+_{\text{in}}(k,\xi)\Omega)}{F_k(\xi)} = 0\,,$$

because of (6). Therefore the expression

$$\frac{B^+_{\text{in}}(k,\xi)}{F_k(\xi)} = B^\pm_{\text{in}}(k)\,,$$

is independent of ξ and

$$B_{\text{in}}(k) = \lim_{\xi\to 0}\frac{B_{\text{in}}(k,\xi)}{F_0(\xi)}\,, \tag{25}$$

does exist because

$$\frac{F_k(\xi)}{F_0(\xi)} = \frac{F(k\xi,\xi^2)}{F(0,\xi^2)}\,,$$

is in the limit $\xi\to 0$ independent of k:

$$\lim_{\xi\to 0}\frac{F_k(\xi)}{F_0(\xi)} = 1\,.$$

From (21), (22), (23) we get:

$$[B^+_{\text{in}}(\boldsymbol{k}), B^-_{\text{in}}(\boldsymbol{p})] = 2k_0^M\delta_3(\boldsymbol{k}-\boldsymbol{p})\,,$$

$$[B^\pm_{\text{in}}(\boldsymbol{k}), B^\pm_{\text{in}}(\boldsymbol{p})] = 0\,,$$

$$[B^\pm_{\text{in}}(\boldsymbol{k}), A_{\text{in}}(y)] = 0\,.$$

Hence

$$[B^{\text{out}}_{\text{in}}(x), B^{\text{out}}_{\text{in}}(y)] = i\Delta(M, x-y)\,, \tag{26}$$

$$[B^{\text{out}}_{\text{in}}(x), A^{\text{out}}_{\text{in}}(y)] = 0\,. \tag{27}$$

Further we have

$$(\Box - M^2)\, B^{\text{out}}_{\text{in}}(x) = 0$$

and

$$(\Omega, B^{\text{out}}_{\text{in}}(x)\Phi_k) = \frac{\exp\,[ikx]}{(2\pi)^{\frac{3}{2}}}, \qquad k^2 + M^2 = 0,$$

if Φ_k is the bound state with four momentum k_μ. $B^{\text{out}}_{\text{in}}(x)$ is given in terms of $A(x)$ by

$$B^{\text{out}}_{\text{in}}(x) = \lim_{\xi\to 0} \frac{1}{F_0(\xi)} \left\{ TA(x+\xi)A(x-\xi) + \int \mathrm{d}x'\, \Delta^{\text{Adv}}_{\text{Ret}}(M, x-x') J(x', \xi) \right\}, \tag{28}$$

$$J(x, \xi) = (\Box_x - M^2) TA(x+\xi)A(x-\xi)\,.$$

According to (25) $B^{\pm}_{\text{in}}(k, \xi)$ is a multiple of $B_{\text{in}}(k)$. From this fact we conclude that the operators $A_{\text{in}}(x)$, $B_{\text{in}}(y)$ together already form an irreducible operator ring. Therefore the whole Hilbert space $\mathcal{H}$ can be built up by the creation operators

$$A^{-}_{\text{in}}(k)\,, \qquad B^{-}_{\text{in}}(k)$$

of incoming or outgoing elementary particles or bound states of momentum k. The state vectors

$$\Phi^{k_1 \dots k_n}_{\text{in}} = A^{-}_{\text{out}}(\boldsymbol{k}_1) \dots A^{-}_{\text{in}}(\boldsymbol{k}_i) B^{-}_{\text{in}}(\boldsymbol{k}_{i+1}) \dots B^{-}_{\text{in}}(\boldsymbol{k}_n)\, \Omega \tag{29}$$

as well as

$$\begin{aligned} &\Phi^{k_1 \dots k_n}_{\text{out}} = A^{-}_{\text{out}}(\boldsymbol{k}_1) \dots A^{-}_{\text{out}}(\boldsymbol{k}_l) B^{-}_{\text{out}}(\boldsymbol{k}_{l+1}) \dots B^{-}_{\text{out}}(\boldsymbol{k}_n)\, \Omega, \\ &(k_i^2 = -m^2 \quad \text{for} \quad i \leqslant l\,, \quad k_i^2 = -M^2 \quad \text{for} \quad i > l) \end{aligned} \tag{30}$$

form a complete orthonormal system of $\mathcal{H}$. The S-matrix is defined as the operator transforming the incoming into the outgoing states:

$$(\Phi^{p_1 \dots p_k}_{\text{in}}, S\Phi^{q_1 \dots q_l}_{\text{in}}) = (\Phi^{p_1 \dots p_k}_{\text{out}}, \Phi^{q_1 \dots q_l}_{\text{in}})\,. \tag{31}$$

Since both systems (29) and (30) form a complete basis of the Hilbert space the S-matrix is unitary.

It is possible to express the S-matrix by the vacuum expectation values of T-products only. To show this we derive from (7) the following reduction formula ([22])

$$[ST(x_1 \dots x_k), A^{\pm}_{\text{in}\alpha}] = \pm i \int \mathrm{d}z f^{\mp}_{\alpha}(z) K^m_z ST(x_1 \dots x_k z) , \tag{32}$$

$$[ST(x_1 \dots x_k), B^{\mp}_{\text{in}\alpha}(\zeta)] = \pm i \int \mathrm{d}z F^{\mp}_{\alpha}(z) K^M_z ST(x_1 \dots x_k, z-\zeta, z+\zeta) , \tag{33}$$

with

$$T(x_1 \dots x_n) = TA(x_1) \dots A(x_n), \qquad \begin{aligned} f^+_\alpha(z) &= f_\alpha(z), \\ f^-_\alpha(z) &= f^*_\alpha(z). \end{aligned}$$

If we now insert plane waves instead of the wave packets $f_\lambda(x)$ we get

$$[ST(x_1 \dots x_k), A^*_{\text{in}}(k)] = -\frac{i\varepsilon(k)}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}z \exp[ikz] K^m_z ST(x_1 \dots x_k z) , \tag{34}$$

$$[ST(x_1 \dots x_k), B^*_{\text{in}}(k)] = \\ = -\frac{i\varepsilon(k)}{(2\pi)^{\frac{3}{2}} F_k(\zeta)} \int \mathrm{d}z \exp[ikz] K^M_z ST(x_1 \dots x_k, z-\zeta, z+\zeta) . \tag{35}$$

Since the state vectors (29) form a complete basis S can be expanded with respect to the incoming fields:

$$S = \sum_{n,m=0}^{\infty} \frac{(-i)^{m+n}}{m!\, n!} \int \mathrm{d}k_1 \dots \mathrm{d}k_m \, \mathrm{d}l_1 \dots \mathrm{d}l_n c(k_1 \dots k_m; l_1 \dots l_n) \cdot \\ \cdot \prod_{i=1}^{m} \delta(k_i^2 + m^2) \prod_{j=1}^{n} \delta(k_j^2 + M^2) : A_{\text{in}}(k_1) \dots A_{\text{in}}(k_m) B_{\text{in}}(l_1) \dots B_{\text{in}}(l_n) : , \tag{36}$$

where the coefficients are

$$\begin{cases} c(k_1 \dots k_m; l_1 \dots l_n) = \\ \quad = \varepsilon(k_1) \dots \varepsilon(k_m) \varepsilon(l_1) \dots \varepsilon(l_n) (\Omega, [\dots [S, A^*_{\text{in}}(k_1)] \dots A^*_{\text{in}}(k_m)] B^*_{\text{in}}(l_1)] \dots B^*_{\text{in}}(l_n)] \Omega) = \\ \quad = \dfrac{1}{(2\pi)^{\frac{3}{2}(m+n)} F_{l_1}(\zeta_1) \dots F_{l_n}(\zeta_n)} \int \mathrm{d}y_1 \dots \mathrm{d}y_m \, \mathrm{d}z_1 \dots \mathrm{d}z_n \exp\left[i\left(\sum k_i y_i + \sum l_j z_j\right)\right] \cdot \\ \cdot K^m_{y_1} \dots K^m_{y_m} K^M_{z_1} \dots K^M_{z_n} (\Omega, T(y_1 \dots y_m, z_1 + \zeta_1, \dots z_n + \zeta_n, z_1 - \zeta_1, \dots, z_n - \zeta_n) \Omega) . \end{cases}$$

([22]) All reduction formulae in H. Lehmann, K. Symanzik and W. Zimmermann: *Nuovo Cimento*, **1**, 425 (1955), Sect. **2** are contained in the single formula (32) which was given by K. Symanzik: unpublished.

The last line of (37) follows by iterating (34), (35) and taking the vacuum expectation value. Transforming (37) into co-ordinate space we get [23]

$$(38)\quad \left\{ \begin{aligned} S = \sum_{m,n=0}^{\infty} \frac{(-i)^{m+n}}{m!\,n!} \lim_{\zeta\to 0} \frac{1}{F_0(\zeta)^n} \int \mathrm{d}y_1 \dots \mathrm{d}y_m \mathrm{d}z_1 \dots \mathrm{d}z_n K^m_{y_1} \dots K^m_{y_m} K^M_{z_1} \dots K^M_{z_n} \cdot \\ \cdot (\Omega, T(y_1 \dots y_m, z_1+\zeta, \dots, z_n+\zeta, z_n-\zeta, \dots z_n-\zeta)\Omega) : \\ : A_{\mathrm{in}}(y_1) \dots A_{\mathrm{in}}(y_m) B_{\mathrm{in}}(z_1) \dots B_{\mathrm{in}}(z_n) : . \end{aligned} \right.$$

The function $F_k(\zeta)$ which enters in the expansions (37) and (38) can easily be expressed by vacuum τ-functions (*i.e.*, vacuum expectation values of T-products) if we take the vacuum expectation value of reduction formula (35) for $k=2$, $x_1=\xi$, $x_2=-\xi$:

$$(39)\quad F_k(\xi)F_k(\eta) = -i\int \mathrm{d}y \exp[iky] K^M_y (\Omega, TA(\xi)A(-\xi)A(y+\eta)A(y-\eta)\Omega) .$$

Putting $\xi=\eta$ we get

$$(40)\quad F_k(\xi)^2 = -i\int \mathrm{d}x \exp[ikx] K^M_x (\Omega, TA(\xi)A(-\xi)A(x+\xi)A(x-\xi)\Omega) .$$

Inserting (40) in (37) or (38) the S-matrix is completely given by the vacuum τ-functions.

4. – Local field operators for bound states.

In this section we want to take the limit $\xi \to 0$ of the operator

$$B(x;\xi) = TA(x+\xi)A(x-\xi)$$

itself. We assume the existence of

$$(41)\qquad B(x) = \lim_{\xi\to 0} \frac{TA(x+\xi)A(x-\xi) - (\Omega, TA(\xi)A(-\xi)\Omega)}{(2\pi)^{\frac{3}{2}}(\Omega, TA(\xi)A(-\xi)\Phi_0)} ,$$

---

[23] One gets corresponding expansions of the T-product replacing S by $ST(x_1 \dots x_k)$ and

$$T(y_1 \dots y_m, z_1+\zeta_1, \dots, z_n+\zeta_n, z_1-\zeta_1, \dots, z_n-\zeta_n) ,$$

by

$$T(x_1 \dots x_k, y_1 \dots y_m, z_1+\zeta_1, \dots, z_n+\zeta_n, z_1-\zeta_1, \dots, z_n-\zeta_n) ,$$

in the expansions (37) and (38).

which in a formal sense can be written as [24]

$$B(x) = a^{-\frac{1}{2}} A(x)^2 - a^{-\frac{1}{2}} b \tag{42}$$

with the two (probably divergent) renormalization constants a and b

$$\begin{cases} a = -i\int \exp\left[-iMx_0\right](\Omega, TA(0)^2 A(x)^2 \Omega)\, \mathrm{d}x_0\,, \\ \\ b = (\Omega, A(0)^2 \Omega)\,. \end{cases} \qquad \text{(according to (40))} \tag{43}$$

The operator $B(x)$ is, of course, invariant under the inhomogeneous Lorentz group and commutes with $B(y)$ and $A(y)$ if $(x-y)^2 > 0$. Furthermore we get from (7) or (28) the asymptotic conditions

$$\lim_{t \to \pm\infty} (\Phi, B_F(t)\Psi) = (\Phi, B^{\mathrm{out}}_{\mathrm{in}_F}\Psi) \tag{44}$$

with

$$B_F(t) = -i\int_{x_0 = t} \mathrm{d}_3 x B(x) \overset{\leftrightarrow}{\frac{\partial}{\partial x_0}} F^*(x)\,,$$

for any normalizable solution of $(\Box_x - M^2)F(x) = 0$.

Therefore the field operators $A(x)$ and $B(x)$ together satisfy the three principles of invariance, causality and asymptotic conditions exactly in the same formulation which was given in an earlier paper for elementary particles only [25]. From these principles we get in the usual way an expansion of the S-matrix:

$$\begin{cases} S = \sum_{m,n=0}^{\infty} \frac{(-i)^{m+n}}{m!\, n!} \int \mathrm{d}y_1 \ldots \mathrm{d}y_m \mathrm{d}z_1 \ldots \mathrm{d}z_n K^m_{y_1} \ldots K^m_{y_m} K^M_{z_1} \ldots K^M_{z_n} \cdot \\ \cdot (\Omega, TA(y_1) \ldots A(y_m) B(z_1) \ldots B(z_n)\Omega) : A_{\mathrm{in}}(y_1) \ldots A_{\mathrm{in}}(y_m) B_{\mathrm{in}}(z_1) \ldots B_{\mathrm{in}}(z_n) :\,, \end{cases} \tag{45}$$

which may also be obtained directly from (38) taking the limit $\zeta \to 0$.

We remark that there remains at least one formal difference between the case of two independent elementary particles and the case of one elementary and one composite particle. If $B(x)$ belongs to a bound state composed of elementary particles described by a field $A(x)$, it is possible to represent $B(x)$

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[24] For the case of A^4-coupling at least in perturbation theory the matrix elements of $B(x)$ can be made finite to any order of the coupling constant if a and b are determined by (43).

[25] H. Lehmann, K. Symanzik and W. Zimmermann: *Nuovo Cimento*, **1**, 425 (1955). Although there only the case of one scalar field was considered the generalization to several fields is obvious.

—roughly speaking—as a polynomial in $A(x)$ (in our model Eq. (42)). We may impose this relationship as an additional condition beyond the principles of invariance, causality and the asymptotic condition in order to exclude the case of two elementary particles.

5. – Examples.

In this Section we shall give simple examples for the expansions of the S-matrix which we have derived in the last sections. Instead of the model of one scalar field we consider the more interesting case of a charged scalar « nucleon » field $\psi(x)$ interacting with a neutral scalar « meson » field $A(x)$. We assume that there are just three non-vanishing discrete eigenvalues m^2, $\varkappa^2$ and M^2 of $-P_\mu^2$: m^2 belongs to the one nucleon states, $\varkappa^2$ to the one meson states and M^2 to a stable bound state Ψ of charge two and spin zero (« deuteron » state). As a consequence of charge conservation

$$(\Omega, \psi(x)\Psi) = 0$$

vanishes identically for a deuteron state Ψ. Under the corresponding assumptions, as formulated in Sect. 1 for the case of one scalar field, we can apply the methods developed in the last sections. As an example for the S-matrix expansions (36), (38) and (45) we consider the special scattering process with one deuteron and one meson of momentum q_μ resp k_μ in the initial state and two outgoing nucleons with momentum p_μ^1 and p_μ^2 in the final state:

$$(36') \quad (\Phi_{\text{in}}^{p_1 p_2}, S\Phi_{\text{in}}^{q,k}) = \frac{\Gamma}{(2\pi)^6 F_q(\zeta)} \int dx_1\, dx_2\, dy\, dz \exp\left[i(qz + ky - p_1 x_1 - p_2 x_2)\right] \cdot \\ \cdot K_{x_1}^m K_{x_2}^m K_y^\varkappa K_z^M (\Omega, T\psi(x_1)\psi(x_2)A(y)\bar{\psi}(z+\zeta)\bar{\psi}(z-\zeta)\Omega) ,$$

$$(38') \quad = \lim_{\zeta \to 0} \frac{\Gamma}{(2\pi)^6 F_0(\zeta)} \int dx_1\, dx_2\, dy\, dz \exp\left[i(qz + ky - p_1 x_1 - p_2 x_2)\right] \cdot \\ \cdot K_{x_1}^m K_{x_2}^m K_z^\varkappa K_y^M (\Omega, T\psi(x_1)\psi(x_2)A(y)\bar{\psi}(z+\zeta)\bar{\psi}(z-\zeta)\Omega) ,$$

$$(45') \quad = \int dx_1\, dx_2\, dy\, dz \exp\left[i(qz + ky - p_1 x_1 - p_2 x_2)\right] \cdot \\ \cdot K_{x_1}^m K_{x_2}^m K_y^\varkappa K_z^M (\Omega, T\psi(x_1)\psi(x_2)A(y)B(z)\Omega) ,$$

$$p_1^2 = p_2^2 = -m^2 , \qquad q^2 = -M^2 , \qquad k^2 = -\varkappa^2 .$$

The function $F_q(\zeta)$ is given by

$$F_q(\zeta) = (2\pi)^{\frac{3}{2}} (\Omega, T\psi(\zeta)\psi(-\zeta)\Phi_q) = (2\pi)^{\frac{3}{2}} (\Phi_q, T\psi(\zeta)\psi(-\zeta)\Omega)$$

and satisfies

$$F_q(\zeta)^2 = -\, i(2\pi)^{\frac{3}{2}} \int \mathrm{d}z \exp\,[iqz]\, K_z^M(\Omega,\, T\psi(\zeta)\psi(-\zeta)\overline{\psi}(z+\zeta)\overline{\psi}(z-\zeta)\Omega)\,.$$

The deuteron field operator $B(z)$ is defined by

$$B(z) = \lim_{\zeta \to 0} \frac{T\psi(z+\zeta)\psi(z-\zeta)}{F_0(\zeta)}\,,$$

where $F_0(\zeta)$ belongs to the deuteron state Ψ_0 at rest. In a formal sense $B(z)$ may be written as

$$B(z) = c^{-\frac{1}{2}}\psi(z)^2$$

with the renormalization constant

$$c = (2\pi)^3(\Omega,\, \psi(0)^2\Psi_0)^2 = -\, i\int \exp\,[-\,iMx_0](\Omega,\, T\psi(0)^2\psi(x)^2\Omega)\,\mathrm{d}x\,.$$

Finally we remark that expressions (36′), (38′) and (45′) hold also if the model contains additional stable bound states. In this case the incoming nucleon, meson and deuteron fields are, of course, not irreducible in the whole Hilbert space, and the S-matrix expansions contain in addition the incoming fields belonging to the higher bound states. But if the state vectors $\Phi_{\text{in}}^{k_1\cdots k_2}$ and $\Phi_{\text{out}}^{p_1\cdots p_2}$ contain only nucleon, meson and deuteron states, expressions for the S-matrix elements $(\Phi_{\text{out}}^{p_1\cdots p_2},\ \Phi_{\text{in}}^{k_1\cdots k_2})$ can be derived from the asymptotic properties of $A_{\text{in}}^{\text{out}}$, $\Psi_{\text{in}}^{\text{out}}$ and $B_{\text{in}}^{\text{out}}$ in a similar way as in Sect. **3**.

* * *

I would like to thank Professor Oppenheimer for the kind hospitality of the Institute for Advanced Study, and I am grateful to the International Co-operation Administration, Washington, for a grant. I am indebted to many physicists in Göttingen, Princeton and Berkeley for helpful discussions.

RIASSUNTO (*)

Si esamina un campo scalare causale e invariante coinvolgente uno stato stabile legato. Si deriva una formula per la matrice S e si dimostra che lo stato legato si può descrivere per mezzo di un operatore di campo locale e invariante. Per semplicità si considera solo il caso di particelle di spin nullo e di stati legati; l'estensione ad altri casi è tuttavia possibile.

(*) *Traduzione a cura della Redazione.*

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The Eigenvalues of a Randomly Distributed Matrix.

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(ricevuto il 4 Agosto 1958)

Summary. — The elements of a symmetric matrix are supposed to be independent random variables, with a distribution law that depends on the distance of the element from the diagonal. What is the distribution of the eigenvalues of this matrix, in the limit as its size becomes infinite? A method, based on the resolution of the symmetric matrix into two triangular ones, is developed, which reduces the problem to a form which has the promise of being soluble by numerical methods.

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1. – In 1953 F. J. Dyson (1) gave a method for the calculation of the distribution function of the eigen-frequencies of a disordered linear chain, in the limit when the chain becomes infinitely long. By a disordered chain is meant « a chain of one-dimensional harmonic oscillators, each coupled to its nearest neighbours by harmonic forces, the inertia of each oscillator and the strength of each coupling being a random variable with a known statistical law. »

An indirect effect of Dyson's method was the catalysis of a spate of papers purporting to generalize his results to three dimensions, mainly with a view to obtain the frequency spectrum of an imperfect crystal. The most recent list of these papers is, to my best knowledge, given by Maradudin, Mazur, Montroll and Weiss (2). Valuable as these works are they are not accurately described as « generalizations » of Dyson's solution, since this was exact whereas most of the more recent works are effectively perturbation treatments.

The work presented here also grew out from a study of Dyson's paper. It is a generalization of that in the sense that an exact « solution » is given

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(1) F. J. Dyson: *Phys. Rev.*, **92**, 1333 (1953).

(2) A. A. Maradudin, P. Mazur, E. W. Montroll and G. H. Weiss: *Rev. Mod. Phys.*, **30**, 175 (1958).

for the frequency spectrum of a chain, whose constituents are coupled to any number of neighbours according to known statistical laws. Since a finite piece of three dimensional lattice can always be « mapped » onto a one dimensional one, this can also be fitted into this scheme provided we choose the law of coupling suitably. Nor is our interest confined to the calculation of the frequency spectrum, but rather we give a programme for calculating the eigenvalues of any symmetric matrix whose elements take values according to some statistical law.

In this connection we have in mind, among other things, the distribution of electronic energies in a disordered crystal, a problem in which, in spite of its importance, only limited progress appears to have been made.

In the above paragraph prudence and honesty compelled us to write *solution* in quotation marks. Stated more exactly the claims of this paper are as follows.

When each of the constituents of a one-dimensional chain is coupled to a small number of its neighbours, the distribution of eigenfrequencies may be derived exactly from the solution of a system of integral equations. The number and complexity of these integral equations grows rapidly with the number of neighbours coupled to each constituent of the chain. These equations are derived by a very simple and general method based on matrix theory. (There is a cryptic remark in Dyson's paper bearing on the possibility of such a method. DEAN ([3]) comments on this remark). In particular, the reader will note how readily some of Dyson's results are obtained in the simplest case of nearest neighbour interaction.

On the other hand, when each constituent is coupled to many others (possibly to all), or in matrix language, when elements far off the diagonal are non-zero, then the analytical solution of the system of very unpleasant integral equations is clearly out of question. Indeed, in this case (and three dimensional problems are of this kind) we had better not speak about a solution, but rather a reduction of the problem to a form amenable to numerica-work. We are, however, quite serious about this aspect of the problem and propose to carry out with God's help detailed and extensive computations as soon as this is feasible. In the meantime a method of procedure is suggested tentatively.

2·1. – We consider the eigenvalues E_i of the $N \times N$ symmetric matrix $\boldsymbol{A}$ (elements $a_{mn} = a_{nm}$), or equivalently, the roots E_i of the determinantal equation

$$0 = \det \boldsymbol{A}'(E) \equiv \prod_{i=1}^{N} (E_i - E) , \tag{1}$$

([3]) P. DEAN: *Proc. Cambr. Phil. Soc.*, **52**, 752 (1956).

where

$$\boldsymbol{A}' = \boldsymbol{A} - E\boldsymbol{I} .$$

In most physical situations that are of interest to us, it is sufficient for the purpose of finding the energy eigenvalues, to consider a real symmetric Hamiltonian matrix, $\boldsymbol{A}$. In the most general case $\boldsymbol{A}$ would be complex and Hermitian. Then our treatment would proceed in much the same way as for a real matrix but at the expense of typographical complications, which would inconvenience printer and reader alike.

As $N \to \infty$ and the eigenvalues E_i are expected to become densely distributed we define from (1) the functions $R(E)$ and $D(E)$ of the variable E,

$$(2) \qquad R(E) \underset{N\to\infty}{=} \frac{1}{N} \log \det \boldsymbol{A}'(E) \underset{N\to\infty}{=} \frac{1}{N} \sum_{i=1}^{N} \log (E - E_i) = \int_{-\infty}^{\infty} \log (E - E') \, D(E') \, \mathrm{d}E' .$$

$D(E)$ thus defined is the density of eigenvalues in the neighbourhood of E. It was shown by Dyson that equation (2) may be inverted so that provided $R(E)$ is known as a function of the complex variable E, $D(E)$ may be found too. This means that we need only apply ourselves to the task of finding $R(E)$ and this is in fact what we propose to do now.

2·2. – The $N \times N$ matrix $\boldsymbol{A}'$ will now be resolved into a product of an $(N \times N)$ lower triangular $\boldsymbol{T}$ and its transpose $\boldsymbol{T}'$,

$$(3) \qquad \boldsymbol{A}' = \boldsymbol{T}\boldsymbol{T}' .$$

The elements t_{rs} ($\neq 0$ for $r < s$) of $\boldsymbol{T}$ are determined by the set of $\frac{1}{2}N(N+1)$ equations

$$(4) \qquad a_{nn} - E = \sum_{r=1}^{n} t_{rn}^2 ,$$

$$(5) \qquad a_{mn} = \sum_{r=1}^{n} t_{nr} t_{mr} , \qquad (n < m).$$

The determination of t_{rs} as a function of E will be seen to be unique by starting at the left upper corner of $\boldsymbol{A}$ and working one's way downwards (and inwards) using at each stage

$$t_{nn}t_{mn} = a_{mn} - \sum_{r=1}^{n-1} t_{mr}t_{nr} \tag{4.a}$$

or

$$t_{nn}^2 = a_{nn} - E - \sum_{r=1}^{n-1} t_{rn}^2 . \tag{5.a}$$

In this line take $\mathrm{Re}\, t_{nn}(E) > 0$ for $E \to -\infty$.]

Now by (3)

$$\det \boldsymbol{A}' = \Big\{\prod_{r=1}^{N} t_{rr}\Big\}^2$$

so that in (2)

$$R(E) = \underset{N\to\infty}{(2/N)} \sum_{r=1}^{N} \log t_{rr}(E) . \tag{6}$$

If we now consider a matrix $\boldsymbol{A}$ whose elements a_{rs} take values according to some distribution law, say $f_{s-r}(a_{rs})$ $(r < s)$ and $g(a_{rr})$, then t_{mn} will also follow a statistical distribution law, say $F_{mn}(t_{mn})$ and $G_{nn}(t_{nn})$. When the size of the matrix increases indefinitely then, as $n \to \infty$, $G_{nn}(t)$ is expected to converge to a value $G(t)$ (*). In this case it is permissible to rewrite (6) as

$$R(E) = 2\int \log t\, G(t)\, \mathrm{d}t . \tag{7}$$

The integration here, as everywhere in the sequel, is over the full range of the distribution function.

The reader is reminded that F_{mn}, G_{nn} and therefore also G are all functions of E through equations (4a) and (5a). In order to avoid crowding of symbols we suppress this dependence of E, which should, however, be borne in mind.

3. – The determination of $G(t)$ will complete, through the use of (2) and (7), our task of finding $D(E)$. First we find G_{nn}, F_{nm}. The quantities on the

(*) This convergence seems natural on physical grounds. Let it be stated, though, that it has not been proved directly, from the equations defining G_{nn}.

right hand side of (4a), (5a) are statistically independent. Therefore

$$F_{mn}(z) = \int\!\dots\!\int \mathrm{d}y\, y\, G_{nn}(y)\, f_{mn}\Big(zy + \sum_{j=1}^{n-1} x_j y_j\Big) \prod_{r=1}^{n-1} \mathrm{d}x_r\, \mathrm{d}y_r\, F_{nr}(x_r)\, F_{mr}(y_r)\,,$$

$$G_{nn}(z) = 2\int\!\dots\!\int z\, g\Big(z^2 + E + \sum_{j=1}^{n-1} x_j^2\Big) \prod_{r=1}^{n-1} \mathrm{d}x_r\, F_{nr}(x_r)\,.$$

When N and $n \to \infty$, then as noted earlier, we assume that $G_{nn} \to G$ and that F_{rs} depends only on the horizontal distance $r - s$ from the diagonal. Also we relabel the distributions F_{rs} of t_{rs} (which have so far been denoted with the upper left corner of the matrix as the origin) as $F^{(r-s)}$ with the point nn as origin of reference. Implicit in this notation is the assumption that as we move away from the diagonal $F_{mn}(y)$ will approach the δ-function $\delta(y)$. The rapidity of this convergence will be dictated by the speed with which the off-diagonal matrix elements a_{ns} will tend to a δ-function distribution as we move away from the diagonal.

The above two equations can then be rewritten as

$$(8)\quad F^{(r)}(z) = \int\!\dots\!\int \dots \mathrm{d}y \cdot y\, f_r\Big(zy + \sum_{j=1}^{\infty} x_j y_j\Big) G(y) \prod_{s=1}^{\infty} \mathrm{d}x_s\, \mathrm{d}y_s\, F^{(s+r)}(x_s)\, F^{(s)}(y_s) \quad (r = 1, 2, \dots)$$

and

$$(9)\qquad G(z) = 2\int\!\dots\!\int \dots z\, g\Big(z^2 + E + \sum_{j=1}^{\infty} x_j^2\Big) \prod_{s=1}^{\infty} F^{(s)}(x_s)\, \mathrm{d}x_s\,.$$

4. – We take a linear chain first. It is understood, we repeat, that f_r will tend, with $r \to \infty$, to a δ-function and in consequence so will $F^{(r)}$, too.

Two cases will now be considered: a) when the constituent interact with the first nearest neighbours and b) when the coupling is with the first and second nearest neighbours.

a) This is Dyson's problem, put in a more abstract forms.

Now

$$f_r(x) = \delta(x) \qquad \text{for } r \geqslant 2.$$

Solutions of the system of equations (8) and (9) are

$$F^{(r)}(x) = \delta(x) \qquad \text{for } r \geqslant 2.$$

Using these solutions we can eliminate $F^{(1)}$ from (8) and obtain from (9) an

integral equation for $G(z)$,

$$G(z) = 2z\iint g(z^2 + E + x^2) f_1(xy) y\, G(y)\, \mathrm{d}y\, \mathrm{d}x\,.$$

This is effectively the integral equation (44) or (40) given by DYSON for particular choices of g or f_1.

b) The solution of (8) and (9) are

$$F^{(r)}(y) = \delta(y) \qquad \text{for } r \geqslant 3.$$

together with the coupled integral equations:

$$F^{(1)}(z) = \iiint\!\!\int f_1(zy + x_1 y_1) f_2(x_1 y')\, G(y)\, G(y')\, F^{(1)}(y_1)\, yy'\, \mathrm{d}x_1\, \mathrm{d}y_1\, \mathrm{d}y\, \mathrm{d}y'\,,$$

$$G(z) = 2z\iiint g(z^2 + E + x_1^2 + x_2^2) f_2(x_2 y')\, G(y')\, F^{(1)}(x_1)\, yy'\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}y'\,.$$

It is clear that the problem rapidly becomes unmanageable by analytical means except for the simplest choice of f_r.

5. – In describing the interactions in a three-dimensional structure (of M^3 constituents) by a matrix, a possible way of ordering is to label the rows or columns corresponding to the $l\boldsymbol{i}+m\boldsymbol{j}+n\boldsymbol{k}$ constituent ($\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ unit vectors; l, m, n integers: 1, 2, ..., M) by

$$r = l + mM + nM^2\,.$$

Let us adopt the same scheme for r of $F^{(r)}$ and f_r occurring in (8) and (9). Then the *meaning* of these equations is clear. Their solution is another matter.

It is fairly clear that as any one of l, m, n, tends to M (which is itself very large), $F^{(r)}$ will tend to a delta distribution, since f_r does so ex-hypothesis. On the other hand, the topology of the coupling in any dimension higher than one effectively excludes the possibility of quasi-analytical solutions of the form *a*) or *b*). What is to be done in this situation?

Evidently, the question of numerical solution must be raised. Now, I think that some adaptation of the Monte-Carlo method to (8) and (9) is probably the best approach, and in fact, intend to explore this avenue further. For the time being I must leave the problem in its present formal, program-

matic form hoping that the method given here will attract the attention of those interested in this field.

* * *

The criticisms of Professor M. H. L. Pryce, F.R.S. and Dr. D. Bohm were of great value in developing this paper.

RIASSUNTO (*)

Si suppone che gli elementi di una matrice simmetrica siano delle variabili casuali indipendenti con una legge di distribuzione dipendente dalla distanza dell'elemento dalla diagonale. Quale è la distribuzione degli autovalori di questa matrice al limite in cui la sua dimensione diventa infinita? Si sviluppa un metodo, basato sulla risoluzione della matrice simmetrica in due matrici triangolari, che riduce il problema a una forma che promette di essere di risolvibile con metodi numerici.

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On the Helium Film in Non-Static Conditions (*).

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(ricevuto il 4 Agosto 1958)

Summary. — The problem of the helium film under non-static conditions is treated in this paper. Conclusions regard essentially the effect of the (subcritical) flow and that of temperature inhomogeneities on the form of the (saturated) film. The existence of a kind of thermomechanical effect of the film is demonstrated. The theory is a substantially phenomenological one, based on the assumption of reversibility. This assumption is briefly discussed in Sect. **4**, while the irreversibile flow is taken into consideration in Sect. **2**.

Introduction.

The purpose of the present paper is to develop a simple, essentially phenomenological, theory of the non-static film (including of course the static film as a particular case).

1. – The isothermal, reversibly flowing film.

The fundamental assumption is that the subcritical flow in helium II is a *reversible* phenomenon. This point has attracted the attention of various authors, F. London in particular (1), and finds its empirical support in various

(*) Part of the present work was the object of a contributed paper at the Kamerlingh Onnes Conference in Leiden, June 1958.

(1) See for instance F. London: *Superfluids*, vol. 2, p. 149-150.

experiments, among which the double-beaker experiment by DAUNT and MENDELSSOHN (2) is a particularly telling one (3).

The assumption of reversibility means that a subcritically flowing film is a system in equilibrium, somewhat as a current-carrying superconductor is. It also implies the introduction of an extra intensive variable describing the state of flow and as such one may choose the flow Φ per cm width of the film (or any equivalent quantity).

The form of the film can be determined by writing the condition that the chemical potential has to be the same everywhere in the system. In particular, it must have the same value at any point in the film as in the bulk liquid.

To find an expression for the chemical potential μ, let us consider the Gibbs function G_{film} referring to an element of the film of unit area and thickness t, at some height z. The volume V of this system is t itself, its mass is $m = \varrho t$ (where ϱ does not differ appreciably from the bulk liquid density, at least for usual, saturated films).

Calling F_{film} the Helmholtz free energy, one has

$$G_{\text{film}} = F_{\text{film}} + pV = F_{\text{film}} + pt\,, \tag{1}$$

where p is the pressure in the vapor phase.

F_{film} is meant to include the contribution ϱtgz of the gravitational potential. On the contrary, it is convenient to write separately the corresponding term in the expression of G_{bulk}, by which symbol we mean the Gibbs function of an equal quantity of helium as considered above but placed at some height z *inside* the bulk liquid. By definition

$$G_{\text{bulk}} = F_{\text{bulk}} + \varrho tgz + p_h t = F_{\text{bulk}} + (\varrho gz + p_h)t\,, \tag{2}$$

where p_h is the hydrostatic pressure in the liquid. Clearly, $\varrho gz + p_h$ is a constant, and taking, as usual, the origin of z at the free surface of the bulk liquid, we have $\varrho gz + p_h = p$ for $z = 0$. Therefore

$$G_{\text{bulk}} = F_{\text{bulk}} + pt\,. \tag{3}$$

Of course, F_{bulk} is simply proportional to the mass ϱt of helium and can therefore be written

$$F_{\text{bulk}} = f_{\text{bulk}}\,\varrho t\,, \tag{3'}$$

where f_{bulk} no longer depends on t.

(2) G. DAUNT and K. MENDELSSOHN: *Nature*, **157**, 839 (1946).

(3) The same experiment has been recently performed with special accuracy by J. F. ALLEN (communication to the Kamerlingh Onnes Conference).

To get the chemical potentials μ_{film} and μ_{bulk}, we have only to resort to the general formula

$$\mu = \left(\frac{\partial G}{\partial m}\right)_{\text{i.v.}}, \tag{4}$$

that is, we have to take the derivatives of expressions (1) and (3) with respect to $m = \varrho t$, holding the intensive variables constant. We get

$$\mu_{\text{film}} = \frac{1}{\varrho}\left(\frac{\partial F_{\text{film}}}{\partial t}\right)_{\text{i.v.}} + \frac{p}{\varrho}, \tag{5}$$

$$\mu_{\text{bulk}} = \frac{1}{\varrho}\left(\frac{\partial F_{\text{bulk}}}{\partial t}\right)_{\text{i.v.}} + \frac{p}{\varrho} = f_{\text{bulk}} + \frac{p}{\varrho}. \tag{5'}$$

By equating we have

$$\frac{1}{\varrho}\frac{\partial}{\partial t}(F_{\text{film}} - F_{\text{bulk}}) = 0, \tag{6}$$

which, apart from the factor $1/\varrho$ [4], is just the equation employed in discussing the profile of the static film in some previous papers [5].

The same equation must hold for the reversibly flowing film, provided the proper changes are made in the expression of F_{film}. These changes amount to the addition of a kinetic energy term

$$E_{\text{kin}} = \tfrac{1}{2}\varrho_s v_s^2 t\,. \tag{7}$$

(Of course, it is just the character of reversibility which ensures that the kinetic energy of the flow makes a purely additive contribution to the *free* energy or the Gibbs function).

This positive term in the expression of G_{film} gives rise—however—to a *negative* one in the expression of μ_{film}. This comes about because by taking the derivative of (7) with respect to ϱt, we have—according to definition (4)—to keep the flow Φ constant [6]. Assuming, as usual, that $\boldsymbol{v}_s$ is the same throug-

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[4] The reason for not dropping the factor $1/\varrho$ is to preserve for the two terms in eq. (6) their homogeneity with the chemical potentials, from which they differ only by the additive term p/ϱ.

[5] S. FRANCHETTI: *Nuovo Cimento*, **4**, 1504 (1956); **5**, 183, 1266 (1957), henceforth referred to as I, II, III. Although not explicitly stated, it is easy to see from the arguments employed (especially in I, p. 1507-8) that F_{film} includes the gravitational contribution, while F_{bulk} does not, as assumed above.

[6] Alternatively, one can say that the equilibrium situation is characterized by the minimum of the Gibbs function (for the *entire* system) under given conditions of p, T *and flow*. The final equations are of course the same as with the procedure followed here.

hout any (horizontal) section of the film, this means a condition

$$v_s t = \text{const} = \frac{\varrho}{\varrho_s} \Phi \,, \tag{8}$$

(expressing Φ in $\mathrm{cm^3(cm\ s)^{-1}}$). Eliminating v_s between (7) and (8), one gets

$$E_{\mathrm{kin}} = \frac{1}{2} \frac{\varrho^2}{\varrho_s} \Phi^2 t^{-1} \,, \tag{7'}$$

from which a contribution

$$\mu_{\mathrm{film}} - (\mu_{\mathrm{film}})_{\mathrm{stat}} = -\frac{1}{2} \frac{\varrho}{\varrho_s} \Phi^2 t^{-2} \,, \tag{9}$$

to expression (5) and to the left hand side of eq. (6) is immediately derived [7].

As is well known, the subcritical flow Φ cannot exceed a maximum value Φ_{max} which is a function of the temperature. We therefore have

$$\Phi = \eta \Phi_{\mathrm{max}} \,, \qquad 0 \leqslant \eta \leqslant 1. \tag{10}$$

On the other hand, the experimental evidence seems to be compatible with the assumption that the maximal flow Φ_{max} is given by

$$\Phi_{\mathrm{max}} = \frac{\varrho_s}{\varrho} \Phi_{\mathrm{lim}} \,, \tag{11}$$

where Φ_{lim} is a limiting value depending only on the kind of the wall.

Through (10) and (11) we get an alternative expression for $\mu_{\mathrm{film}} - (\mu_{\mathrm{film}})_{\mathrm{stat}}$, namely

$$\mu_{\mathrm{film}} - (\mu_{\mathrm{film}})_{\mathrm{stat}} = -\frac{1}{2} \frac{\varrho_s}{\varrho} \eta^2 \Phi_{\mathrm{lim}}^2 t^{-2} \,. \tag{9'}$$

We have therefore as equation determining t in the presence of a flow

$$-\frac{1}{2} \frac{\varrho_s}{\varrho} \eta^2 \Phi_{\mathrm{lim}}^2 t^{-2} + \frac{1}{\varrho} \frac{\partial}{\partial t} (F_{\mathrm{film}} - F_{\mathrm{bulk}})_{\mathrm{stat}} = 0 \,. \tag{12}$$

[7] We have entered these details, which seem fairly plain, because they seem to have been overlooked by some author. In particular one should carefully distinguish between the kinetic contribution to φ and the specific kinetic energy, the former being just the negative of the latter.

The diffrence $F_{\text{film}} - F_{\text{bulk}}$ in the static case has been carefully investigated in the papers quoted in ref. (5). It is due on the one hand to the fields acting on the film (Van der Waals field of the wall, gravitational field), while on the other hand there are terms due to the smallness of the transverse dimension of the film. The latter are two: a temperature independent term relating to the zero point energy of the liquid, regardless of its state of excitation (« Atkins' term ») and a temperature dependent one due to statistic effects on the excitations themselves. These « limitation » terms are the only ones requiring atomistic considerations to be worked out. The choice of the model employed should however have only secondary importance.

The limitation terms in $F_{\text{film}} - F_{\text{bulk}}$ contribute to the left hand side of (12) (or to the expression of μ_{film}) terms in t^{-2}, while the Van der Waals term is one in t^{-3} and the gravitational potential is of course simply gz.

For details the reader is referred to the papers quoted in ref. (5). However, the Van der Waals term can immediately be derived—in a somewhat simplified way—by noting that the Van der Waals energy per cm^2 of the film will have the form

$$-\int_{x_0}^{t} a x^{-3}\,\mathrm{d}x\,,$$

where x is the distance from the wall, x_0 some (inessential) minimal distance of approach and a is a positive constant depending on the nature of the wall. The contribution to μ_{film} due to the above term is its derivative with respect to ϱt, that is

$$-\frac{a}{\varrho}\,t^{-3}\,. \tag{13}$$

If A is the constant in the expression $-Ax^{-3}$ for the potential energy of a helium atom at a distance x from the (plane) wall, a/ϱ turns out to be $(1/m)(A - 2A_{\text{He}})$ with m the mass of the helium atom and A_{He} characterizing the action of a fictitious « helium wall » (having the density of liquid helium). In other words

$$a = \frac{\varrho}{m}\,(A - 2A_{\text{He}}) = 2.19 \cdot 10^{22}\,(A - 2A_{\text{He}})\,. \tag{13'}$$

Some approximate values for A calculated according to SCHIFF (8) are listed in Table I (units are c.g.s.).

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(8) L. I. SCHIFF: *Phys. Rev.*, **59**, 839 (1941).

Referring for instance to eq. (19) of paper III, one finds

$$\frac{1}{\varrho}\frac{\partial}{\partial t}(F_{\text{film}} - F_{\text{bulk}})_{\text{stat}} = -\frac{1}{\varrho}(at^{-3} + bt^{-2}) + gz\,, \tag{14}$$

with a given by eq. (13) and

$$b = [1.87 + R(T)]\cdot 10^{-9} \qquad \text{(c.g.s. units).} \tag{14'}$$

TABLE I.

Copper	$A = 5.3\cdot 10^{-37}$
Glass	4.3 »
Paraffin wax	1.7 »
Solid air	0.97 »
Solid hydrogen or neon . . .	0.5 »
« Liquid helium »	$A_{\text{He}} = 0.17$ »

The detailed form of $R(T)$ is rather laborious to get and can be deduced in two instances from paper III. Table II shows some approximate values for a copper wall.

TABLE II.

T	$R(T)$	T	$R(T)$
$\leqslant 1.0$	negligible	2.00	0.82
1.20	0.017	T_λ	1.49
1.40	0.064	2.20	1.52
1.60	0.17	2.50	1.56
1.80	0.40	3.00	1.47

Inserting expression (14) into eq. (12) one has

$$-\frac{1}{2}\frac{\varrho_s}{\varrho}\eta^2\Phi_{\text{lim}}^2 t^{-2} - \frac{1}{\varrho}(at^{-3} + bt^{-2}) + gz = 0\,,$$

or else

$$a + (b + \tfrac{1}{2}\varrho_s\eta^2\Phi_{\text{lim}}^2)t = \varrho g z t^3\,. \tag{15}$$

Fig. 1 suggests a graphical way of solving this equation with respect to t (by given z). It also shows an approached but quick evaluation of the flow

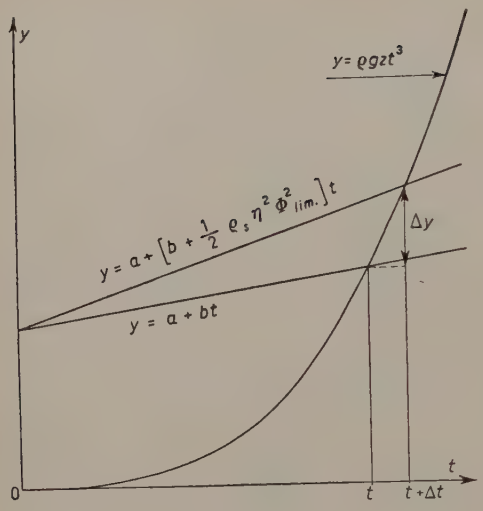

Fig. 1.

effect. Indeed one has

$$\frac{\Delta y}{\Delta t} \approx 3\varrho g z t^2, \qquad \Delta y \approx \frac{1}{2}\varrho_s \eta^2 \Phi^2_{\text{lim}} t,$$

and therefore

$$\frac{\Delta t}{t} \approx \frac{\varrho_s \eta^2 \Phi^2_{\text{lim}}}{6\varrho g z t^2}. \tag{16}$$

Visibly, the effect is an *increase* of the thickness for the flowing film. Intuitively, this is comprehensible, because while thickening the film *decreases* its free energy by reducing the amount of kinetic energy associated with the given flow. (If attention is not paid to the considerations in footnote (7), the opposite result obtains.)

Since zt^2 is an increasing function of z, the effect is larger at lower heights. It is also larger at lower temperatures, because of the factor ϱ_s/ϱ which, moreover, makes the effect vanish at the λ-point.

In most experiments the flow has its « saturation » value, that is $\eta = 1$, while the temperature is sufficiently low to have $\varrho_s/\varrho \approx 1$. A typical value for Φ_{lim} on a clean metal or glass surface is $\sim 10^{-4}$. With $zt^2 \sim 2 \cdot 10^{-11}$ which is also a typical value in ordinary conditions, one sees from (16) that an effect of the order of 8% can be expected.

2. – On the irreversibly flowing film.

In all rigour, the treatment of the foregoing Section refers to a truly reversible flow, such as takes place on the walls of the inner beaker in the double-beaker experiment (flow without level difference). The situation is however somewhat different when the film is flowing from a higher to a lower level, as sketched in Fig. 2.

Here the chemical potential of the liquid to the left is higher than that to the right by an amount $g(z_0 - z_A)$. This means that the chemical potential cannot have the same value all along the film.

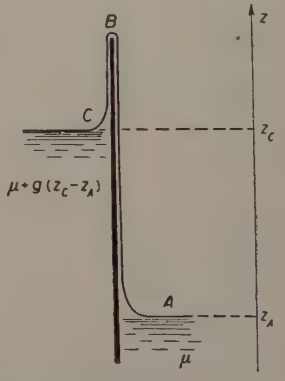

Fig. 2.

If $z_c - z_A$ is sufficiently small, the loss of potential energy can be counterbalanced by a gain of kinetic energy without exceeding the limits of the subcritical flow: this is what happens in oscillating film experiments. Excluding this case and considering instead a *stationary* flow in the conditions just described, we have clearly to do with an irreversible phenomenon in which the potential energy $g(z_c - z_A)$ per unit mass of the flowing liquid must eventually be turned into heat. (This does not mean that the flow is non-isothermal, since—presumably—that small amount of heat will get absorbed by the wall).

One feels, however, that at least in regions away from point B, the form of the film should be determined by the amount of flow, independently from any other circumstance accompanying it.

Thus, one is led to think that the irreversible aspect of the flow is probably confined to a « transition zone » around point B, where the change of μ (which will not be definable inside this region) and the « matching » of the different thicknesses of the ascending and descending film are to take place.

Indeed, in that region there will be a « squeezing » of the flow, that is an acceleration of the fluid. This might well be the cause for friction setting in and causing the conversion of mechanical energy into heat. (It is hard to see how this could happen in the lower regions, where the thickness of the film is continually increasing).

A proof that the squeezing is an important factor may be seen in the instability that arises when the difference in the thickness on the two sides is comparatively larger, that is when the upper level lies sufficiently near to the rim, leading to drop formation and increasing the flow above its normal value ([9]).

3. – The non-isothermal (resting or flowing) film.

In the non-isothermal case too, we shall require the chemical potential to be the same everywhere and hence

$$d\mu = 0 , \tag{17}$$

all along the film, which means that we shall treat the film as being in equilibrium. (This point is briefly analized in the next section).

It is hardly necessary to recall that the same assumption is the basis of a well known method of deriving the fundamental equation for the thermomechanical effect between two vessels connected by a superleak ([10]). The dif-

([9]) L. C. Jackson and D. G. Henshaw: *Phil. Mag.*, **41**, 1081 (1950).

([10]) See F. London: *Superfluids*, vol. **2**, p. 70. Or else, L. Tisza: *Phys. Rev.*, **72**, 845 (1947).

ference is that in the latter case a pressure gradient is originated as a consequence of a temperature gradient. In the film this is impossible, and in place of a pressure gradient we shall find a *deformation* of the film, which may also be accompanied by a flow and constitutes a sort of isobaric thermomechanical effect of the film.

The chemical potential μ at any point of the film can be thought to be a function of the variables t, z, p, T and $\Phi = \eta(\varrho_s/\varrho)\Phi_{\text{lim}}$. We shall consider the (external) pressure p to be a constant, and for simplicity's sake we shall assume the same to hold for the flow Φ. (Although in the presence of some evaporation along the film this may not be the case).

Eq. (17) becomes therefore

$$\frac{\partial \mu}{\partial t}\,\mathrm{d}t + \frac{\partial \mu}{\partial z}\,\mathrm{d}z + \frac{\partial \mu}{\partial T}\,\mathrm{d}T = 0\,.$$

The third differential coefficient can be replaced by the negative of the average specific entropy $-s$, getting

$$\frac{\partial \mu}{\partial t}\,\mathrm{d}t + \frac{\partial \mu}{\partial z}\,\mathrm{d}z - s\,\mathrm{d}T = 0\,. \tag{17'}$$

The appropriate expression for μ can be obtained from eq. (5), combining it with eqs. (5′) and (14) and adding the flow contribution (9′). The result is

$$\mu = -\frac{1}{\varrho}\,a t^{-\frac{1}{3}} - \frac{1}{\varrho}\left(b + \frac{1}{2}\,\varrho_s \eta^2 \Phi_{\text{lim}}^2\right) t^{-2} + gz + \frac{p}{\varrho} + f_{\text{bulk}}\,. \tag{18}$$

The last two terms give no contribution to eq. (17′) which becomes

$$-\frac{1}{\varrho}\,\frac{\partial}{\partial t}\left[a t^{-\frac{1}{3}} + \left(b + \frac{1}{2}\,\varrho_s \eta^2 \Phi_{\text{lim}}^2\right) t^{-2}\right]\mathrm{d}t + g\,\mathrm{d}z - s\,\mathrm{d}T = 0\,. \tag{19}$$

By integrating with respect to variables t and z from $z = 0$ (where $t = \infty$) up to any height z, one has

$$-\frac{1}{\varrho}\,a t^{-\frac{1}{3}} - \frac{1}{\varrho}\left(b + \frac{1}{2}\,\varrho_s \eta^2 \Phi_{\text{lim}}^2\right) t^{-2} + gz - \int_0^z s\,\frac{\partial T}{\partial z}\,\mathrm{d}z = 0\,.$$

As the temperature differences will always be very small, the last term may conveniently be written $\bar{s}(T - T_0)$ with $\bar{s}$ an average value and T_0 the temperature at $z = 0$.

Written in a form analogous to (15), the above equation becomes therefore

$$a + \left(b + \frac{1}{2}\varrho_s \eta^2 \Phi_{\lim}^2\right) t = \varrho g \left[z - \frac{\bar{s}}{g}(T - T_0)\right] t^3 , \tag{20}$$

which reduces to eq. (15) when $T = T_0$ troughout.

It follows from this equation, comparing with eq. (15), that a situation with $T > T_0$ corresponds to have z changed by an amount

$$\Delta z = -\frac{\bar{s}}{g}(T - T_0) , \tag{20'}$$

meaning an *upward sliding* of the film. If instead one has $T < T_0$ a corresponding *downward sliding* takes place, both displacements being in accordance with the character of a thermomechanical effect (displacement towards the warmer region).

The kinship of the present effect with the ordinary thermomechanical effect is also apparent when one considers the well known experiment by DAUNT and MENDELSSOHN [11] in which two vessels containing liquid helium at two slightly different temperatures are connected through the film. Were it not for the distillation which tends to counteract the effect (see next section) a situation of equilibrium would be reached with the liquid at two different levels in the vessels. Eq. (19) would apply. By integrating all along the film from the one vessel to the other, the t-containing terms vanish and we find

$$g(z_2 - z_1) = \bar{s}(T_2 - T_1) .$$

Since, obviously, $\varrho g(z_2 - z_1)$ gives the « pressure head » ΔP, we have actually found the well known equation of the fountain effect $\Delta P = \varrho \bar{s} \Delta T$.

It follows also from eq. (20) that t would become infinity and then negative if the square bracket in the right hand side should turn respectively zero or negative. This means that to have a stationary film, the condition

$$\bar{s}(T - T_0) < gz \tag{21}$$

or else

$$\frac{\mathrm{d}T}{\mathrm{d}z} < \frac{g}{s} , \tag{21'}$$

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[11] J. G. DAUNT and K. MENDELSSOHN: *Proc. Phys. Soc.*, **63**, 1305 (1950).

must be fulfilled. If this does not happen, that is if the (upward) temperature gradient is too large, *instability* will set in. Instability in the presence of a temperature gradient was reported (and correctly interpreted as a sort of thermomechanical effect) by HAM and JACKSON ([12]).

Directly from eq. (19) one may derive an expression for $\mathrm{d}t/\mathrm{d}z$:

$$\frac{\mathrm{d}t}{\mathrm{d}z} = -\left[3at^{-4} + 2\left(b + \frac{1}{2}\varrho_s\eta^2\Phi_{\mathrm{lim}}^2\right)t^{-3}\right]^{-1}\varrho\left(g - s\frac{\mathrm{d}T}{\mathrm{d}z}\right). \tag{22}$$

We can see from here and from (21′) that dt/dz is always *negative*, that is, we cannot have a «bulging» (stationary) film. Eq. (22) shows, however, that $\mathrm{d}t/\mathrm{d}z$ can reach (negative) values as small as we please when $s(\mathrm{d}T/\mathrm{d}z)(>0)$ is made to approach «from below» the value of g.

This might account for the large values found in some experiments ([13]) for the parameter q in the expression $t = \mathrm{const}\ z^{-1/q}$ for the thickness of the film, if we note that

$$\frac{1}{q} = -\frac{z}{t}\frac{\mathrm{d}t}{\mathrm{d}z}. \tag{22′}$$

The observation following eq. (20), that the thickness of the non-isothermal film is the same as for the isothermal film at a point shifted vertically by an amount Δz given by eq. (20′) suggests the following approached expression for the thickness change Δt due to temperature inhomogeneities

$$\Delta t \approx \left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)_{\mathrm{iso}}\cdot\Delta z = -\left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)_{\mathrm{iso}}\cdot\frac{\bar{s}}{g}(T - T_0).$$

Combining this with (22′) one gets

$$\frac{\Delta t}{t} \approx \left(\frac{1}{q}\right)_{\mathrm{iso}}\cdot\frac{1}{z}\frac{\bar{s}}{g}(T - T_0). \tag{23}$$

If Δt is negative (*i.e.* $T < T_0$), it cannot of course exceed t, so that

$$\left|\frac{\Delta t}{t}\right| < 1. \qquad (\Delta t < 0). \tag{23′}$$

If Δt is positive ($T > T_0$), $|\Delta t/t|$ is also bounded because of condition (21)

([12]) A. C. HAM and L. C. JACKSON: *Proc. Roy. Soc.*, **240**, 243 (1957).
([13]) See for instance K. R. ATKINS: *Proc. Roy. Soc.*, **203**, 123 (1950).

which imposes

$$\left|\frac{\Delta t}{t}\right| \leqslant \left(\frac{1}{q}\right)_{\text{iso}} . \qquad (\Delta t > 0). \tag{23''}$$

Since, normally, in the isothermal case $q \approx 2.5 \div 3.0$, we see that the relative thickness increase cannot exceed $(30 \div 40)\%$.

The average temperature gradient $(T - T_0)/z$ associated with a given relative change $\Delta t/t$ follows immediately from (23). In Table III are listed some values for $\Delta t/t = 0.1$. They show the great sensitiveness of the film to temperature disturbances, in the upper part of the He II region ([14]).

TABLE III.

T (°K)	grad T (°K/cm)	T (°K)	grad T (°K/cm)
0.0	∞	1.5	$1.4 \cdot 10^{-4}$
0.5	$2.7 \cdot 10^{-2}$	2.0	$2.9 \cdot 10^{-5}$
1.0	$1.6 \cdot 10^{-3}$	T_λ	$1.7 \cdot 10^{-5}$

Suppose now that in some part of the film the maximum relative increase of the thickness has been reached. This means, according to (22) and (23), $\mathrm{d}t/\mathrm{d}z \approx 0$, that is the film surface is nearly parallel to the wall. Suppose moreover that in a neighbouring (higher) region Δt falls sharply. Then a *step-like* profile results for the film. (A temperature distribution giving an effect of this kind would, for instance, be a « peaked » one, such as could be originated by some heat radiation falling on a narrow strip across the film).

Steps in the profile of the film have been reported by DILLINGER ([15]) and by HAM and JACKSON (ref. ([12])).

If Δt is to fall from its maximum value to zero in a range of the order of one cm, it is easily found from eq. (23) and the known values of s, that $\mathrm{d}T/\mathrm{d}z$ must take values ranging from 10^{-2} to 10^{-4} °K/cm (depending on the values of T and z).

4. – On the nature of the equilibrium in a stationary film.

The foregoing treatment has been based on the assumption that a stationary film is a system in equilibrium. That in certain instances this has to be considered as a convenient schematization rather than a rigorous description, is

([14]) The entropy values utilized in the computation have been taken from C. F. SQUIRE: *Low Temperature Physics* (New York, 1953), p. 64. The very small difference between s_{bulk} and s_{film} is of course of no significance in the present connection.

([15]) J. R. DILLINGER: *Conf. de Physique des Basses Températures* (Paris, 1955), p. 97.

shown by DAUNT and MENDELSSOHN's experiment, ref. ([11]), where the two masses of helium would have to be thought of as being in equilibrium insofar they are connected through the film, whereas they show manifestly to be not when the exchange of helium vapour is considered. Indeed, in a general way it may well look surprising that we can speak at all of equilibrium between bodies having different temperatures.

The following considerations should help clarify the matter.

The key to the paradox lies in the observation that we would presumably have a truly reversible process in a transfer of superfluid *alone*. Indeed, it is conceivable that the change from rest to motion—as long as the flow remains subcritical—reduces to an adiabatic (in the quantum mechanical sense) alteration of the fundamental state eigenfunction, requiring no rearrangement in the occupation of states and hence no entropy change. The same however could not be said of any transfer of the *normal* fluid (let alone of the vapour).

There would therefore be true equilibrium between two helium bodies connected *only* via a channel rigorously impervious to the normal fluid. However, apart from the fact that it would be often impossible to avoid other connections (as that via the vapour in the previous example), the above condition would always mean an ideal case. Indeed, through any real channel the normal fluid would have to move, too, though much more slowly, bringing eventually the system to « ordinary » equilibrium, in particular, to isothermal conditions.

We therefore see that the treatment given in the previous sections is an approximation valid to the extent that the transfer of the normal fluid (or the vapour) is negligible against that of the superfluid. In many instances, this may well represent a very good approximation.

RIASSUNTO

Questo lavoro concerne il comportamento del film (saturo) dell'elio in condizioni non statiche. Viene studiato principalmente l'effetto del movimento (flusso subcritico) e quello di inomogeneità di temperatura sulla forma del film. Si dimostra l'esistenza di una sorta di effetto termomeccanico del film. La teoria è essenzialmente fenomenologica e si basa sull'ipotesi della reversibilità dei fenomeni. La legittimità di questa ipotesi viene esaminata nella Sez. 4, mentre il caso del flusso irreversibile viene preso in considerazione nella Sez. 2.

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A Decay of an Anti-hyperon, $\overline{\Lambda^0}$, in Nuclear Emulsion.

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(ricevuto il 4 Agosto 1958)

Summary. — An event which has been found in a stack of nuclear emulsion exposed to 10^6 4.6 GeV π^--mesons per cm^2, has been interpreted as an example of the decay in flight of an anti-hyperon, $\overline{\Lambda^0}$. The event and the interpretation are discussed in detail.

1. – Introduction.

The concept of particle and anti-particle is now well established and it received experimental confirmation a few years ago with the discovery of the anti-proton and anti-neutron ([1,2]). Up until now however it has not been extended to the realm of the hyperons. As distinct from the other group of strange particles, the K^+ and K^--mesons which are particle and anti-particle, the presently known hyperons are expected to have anti-particles which have not yet been discovered. They are however expected to exist and will fit naturally into the present scheme of elementary particles. The main experi-

(*) On leave from the University of Bristol.

(+) Partially supported by the United States Atomic Energy Commission.

(1) O. Chamberlain, E. Segrè, C. Wiegand and T. Ypsilantis: *Phys. Rev.*, **100**, 947 (1955).

(2) B. Cork, G. R. Lambertson, O. Piccioni and W. A. Wenzel: *Phys. Rev.*, **104**, 1193 (1956).

mental difficulty in observing such particles lies in the fact that they must be produced in association with a baryon to conserve the baryon number or must be produced by an anti-baryon in such a way as to conserve strangeness, the threshold energy for production is therefore very high. The only hyperon-antihyperon pair whose threshold energy for direct production is within reach with accelerators presently working is that of the Λ^0-hyperon. The reaction giving an anti-Λ^0 with protons is expected to be of the form:

$$p + n \rightarrow \Lambda^0 + \overline{\Lambda^0} + p + n$$

and with π^--mesons of the form:

$$\pi^- + p \rightarrow \Lambda^0 + \overline{\Lambda^0} + n\,.$$

The threshold energy for production from a free nucleon by a proton is 7.10 GeV and for a π^--meson, 4.73 GeV in the laboratory system; from a bound nucleon it can be considerably less if the struck nucleon is moving in a favourable direction. If a composite nucleus is used as a target however a fraction of $\overline{\Lambda^0}$ will be absorbed before they can escape from the parent nucleus. When intense beams of π^--mesons of ~ 5.0 GeV became available at the Bevatron, it was decided to attempt the detection of an anti-Λ^0-hyperon in nuclear emulsion despite the numerous difficulties involved, of which the last mentioned is one. Because the beam energy is so little above the threshold, the particles produced are ejected in a narrow forward cone in the laboratory system. It is therefore necessary to search for these particles against a large background of beam tracks; it is thus highly desirable to have a pencil beam with a sharp cut-off in one dimension at least. The anti-hyperons are expected to have a short lifetime, $\sim 10^{-10}$ s, and it is therefore necessary to detect them close to the point of production. A convenient way of doing this in nuclear emulsion is to use it as a target and as a detector, particles with lifetimes as low as 10^{-15} s can then be detected. If this technique is used however no separation of the secondary particles is possible either in momentum or mass, one is forced to detect all particles produced. There are hopes that very high fields which can be pulsed with the accelerator may eventually overcome these difficulties.

2. – Exposure details.

The stack size was 150 plates 8 in. $\times$ 7 in. $\times$ 600 μm G.5 Ilford emulsion. The top edge of the stack was exposed to 10^6 π^--mesons/cm^2 which is a density which does not allow one to follow through near-minimum tracks although

it is possible to search for double stars etc. The density of interactions is about 1 per field of view under a magnification of $\times 600$. The density of tracks falls off to $10^3/cm^2$ in a vertical distance of 8 in. The stack was exposed with the individual pellicles vertical as the fall off in this direction was much greater than in the horizontal direction. The mean beam momentum was 4.63 GeV/c at the centre line of the stack (plate 75). The circle of confusion, of radius 3 in; was considerably greater than the dispersion of the system, 0.13 GeV/c per inch. The maximum momentum passed by the analysing system was 5.0 GeV and this momentum would just reach the first 90 plates in the stack. The beam momentum in the first plates was therefore a rather homogeneous mixture between 5.0 and 4.47 GeV/c with a peak at 4.86 GeV/c, or in terms of energies, between 4.85 and 4.32 GeV with a peak at 4.6 GeV. We thus have π^--mesons in the stack whose energies are above threshold for $\overline{\Lambda}^0$-production from a free nucleon.

3. – Experimental procedure.

The detection of neutral particles in emulsion is always difficult and in this case a method first used by FRIEDLANDER *et al.* [3] to find Λ^0-hyperons, that of following back secondary π-mesons, was adopted. The Λ^0-hyperons are expected to decay into positive π-mesons and to antiprotons:

$$\overline{\Lambda}^0 \to \pi^+ + \overline{p} + 37.45 \text{ MeV}.$$

The hyperons will be emitted fast and forward in the laboratory system, the π-mesons will therefore be emitted predominantly in the forward cone. The following method of scanning was therefore adopted: typical π-μ-e decays were found by area scanning in a region of the plate where they could have more probably been produced by the decay of a $\overline{\Lambda}^0$-hyperon. They had to appear to originate in the intensely irradiated part of the stack and they had to make an angle of less than 60° with the primary beam direction. 392 π^+-mesons were followed back until their parent stars were reached or until they were lost in the very high background of tracks. Unfortunately about half of the π-mesons met the latter fate after a mean range of 2.5 cm. The rest were traced to their parent stars and of these 150, 127 appeared to come from primary 4.6 GeV π-meson stars as judged from the direction of a possible primary. Three π-mesons appeared to originate from the decays of K^+-mesons. The kinematics of one event were consistent with a $K^+_{\pi 2}$ decay

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[3] M. W. FRIEDLANDER, D. KEEFE, M. G. K. MENON and M. MERLIN: *Phil. Mag.*, **45**, 533 (1954).

in flight and the other events were consistent with the decay of a $K^+_{\pi 3}$ at rest, giving secondaries 2.01 and 1.5 cm long. The event of interest however is a « V » type event at which a π-meson of 32 MeV originated. In the following sections it will be shown that this event is the decay of an anti-lambda hyperon and probably represents the first example of its kind detected.

4. – Description of the event.

The position of the event in relation to the stack is shown in Fig. 1. The decay event was found 2.7 cm from the highly irradiated stack edge and 4.2 cm from the front of the stack. The direction of the $\overline{\Lambda^0}$ can be inferred exactly from a measurement of the direction and energies of the secondaries. The line of flight makes an angle of 21° with the π^--meson beam direction. The potential path length of the $\overline{\Lambda^0}$ in the stack is such that assuming the normal Λ^0 lifetime for the $\overline{\Lambda^0}$, it was probably produced in the stack. Although an attempt was made to locate the parent star, it was unsuccessful owing to the very large background and high star density. A drawing of the event is shown in Fig. 2.

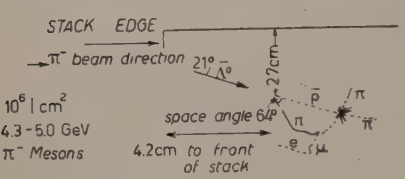

Fig. 1.

The π^+-meson was followed for a distance of 1.70 cm from the π-μ decay point to the « V » event. The π-μ-e event was found 126 μm from the glass interface and the π-meson traversed 5 plates from its origin in plate 46, 290 μm from the air interface to its decay point. The μ-meson range was 621 μm and the decay electron was detected, the π-μ-e event is therefore perfectly normal. The angle of dip of the π-meson was 37° at the « V » event, the meson travelling down in the stack. The other track from the « V » travelled up in the stack at quite a shallow angle of 6°. It traversed 5 plates before interacting in flight 2.71 cm from the « V ». The interaction is very large having 11 prongs, three of which are shower tracks. The projected angle in emulsion between the π and the $\overline{p}$ is 51°, making the space angle $(64 \pm 1)°$. The ionization of the anti-proton track is about twice minimum at the decay point and it changes by 10% over the length of 2.71 cm. As this change of ionization is very critical for the interpretation of the event, it is treated at length in the next section. To prove that the $\overline{\Lambda^0}$ is the correct interpretation of this event, it is necessary to show the identity and direction of both decay particles and to show that the Q-value is that expected from observations on normal Λ^0-hyperons. Even then there is one possibility which cannot be completely ruled out, namely that the decay event is the interaction in flight of

Fig. 2.

an anti-neutron producing a single π-meson and an antiproton with an apparent Q-value in agreement with that for a Λ^0. Such a radical explanation is certainly unlikely and we ignore it.

5. – The anti-proton track.

As mentioned above the direction of this track is vital for the interpretation. Unfortunately only 2.7 cm of range are available for measurement and the change in ionization is only 10% over this distance. Very careful measurements are therefore needed to detect the 10% unambiguously. The track is completely contained in 5 plates. It starts 290 μm from the air interface in the first plate and ends 64 μm from the air interface in the 5-th plate. In Table I are listed the corrected ranges in each plate and the portion of the total range on which ionization measurements are possible (excluding 50 μm from the surface and glass interfaces and other regions of obvious distortion or bad local development). The effective ranges from the « V » event at which each ionization measurement was made are also listed together with the actual number of grains counted and the values of the minimum ionization in each plate. The value of minimum ionization was determined in the following way: the ionization of 4.6 GeV π-mesons is « plateau » and these beam tracks therefore provide a very good means for normalizing each plate. There are a very large number present and it is therefore possible to normalize the exact portion of the emulsion which is actually required. In each plate 10 beam tracks were taken in the same region of the emulsion as the anti-proton track and 1000 grains were counted on each.

TABLE I.

Plate	Total corrected range	Range counted	No. grains	Minimum	g^*	Effective range from $\overline{\Lambda^0}$ decay
1 (46)	2.78 mm	2.27	718	15.00	2.104	1.13 mm
2 (45)	5.88	4.72	1572	15.51	2.150	5.72
3 (44)	6.59	5.11	1780	15.62	2.230	11.98
4 (43)	6.78	5.42	1890	15.39	2.260	18.67
5 (42)	5.09	4.56	1601	15.19	2.316	24.84

The top and bottom 50 μm were excluded. The results showed a gradient of 10% from the top to the bottom of the emulsion. For each plate a mean value was deduced by averaging measurements taken at equal intervals in depth and this is the value (after conversion to minimum from plateau) which appears in Table I. The accuracy of each value is $\pm$ 1% and the deviations

from a common mean are therefore significant. ALEXANDER and JOHNSTON [4] give a plateau minimum ratio of 13%. Previous determinations [5,6] have not given such a high value (about (6÷10)%). However as we have adopted the calibration curves given by ALEXANDER and JOHNSTON, which are normalized to « minimum », we have used their value for this ratio. We have merely reduced our plateau value by 11.5% to give the minimum values. It is realized however that the lack of a precise knowledge of this ratio introduces a systematic error in the determination of the energy of the antiproton although it does not affect the determination of its direction. On the experimental evidence it is unlikely that the ratio exceeds 13% but it is possible that it is as low as 6%. We have therefore included an asymmetric error in the ionization of $^{+7}_{-0}$%.

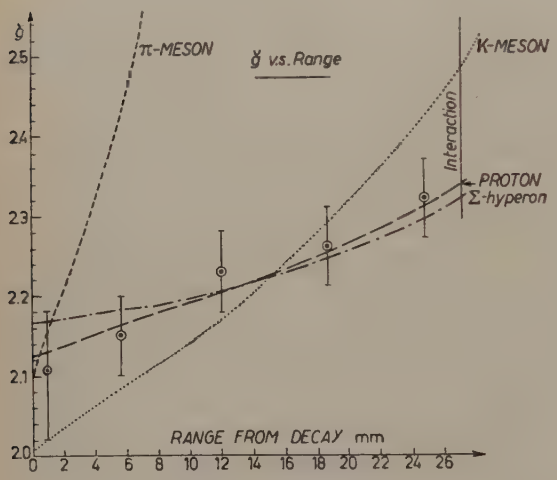

Fig. 3.

The results obtained are shown in Fig. 3. It is clear that the track is moving away from the decay point and towards the large star. The lines shown in the figure are the best fits to the points for various particles. It is patent that the track cannot be that of a π-meson. The points are bad a fit to the K-meson line, but probably equally good to the proton or hyperon line although the proton fit is slightly favoured. It is not possible to distinguish the proton and hyperon masses by these ionization measurements. The fit to a hyperon going in the reverse direction is exceedingly bad but if one admits the reverse direction, the track is probably a hyperon track. These measurements give a value of $g^* = 2.34$ at the point of decay and assuming a protonic mass, correspond to a residual range of 10.0 cm and an energy of 230 MeV. The error without the normalizing error is $\pm$ 5 MeV; with, it is $^{+22}_{-7}$ MeV.

The track is long, and flat enough to permit a measurement of the multiple scattering. This does not give a result of sufficient precision to determine

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(4) G. ALEXANDER and R. H. W. JOHNSTON, *Nuovo Cimento*, **5** 263 (1957).

(5) A. H. MORRISH: *Phil. Mag.*, **43**, 533 (1952); *Phys. Rev.*, **91**, 423 (1953).

(6) R. R. DANIEL, J. H. DAVIES, J. H. MULVEY and D. H. PERKINS: *Phil. Mag.*, **43**, 753 (1952).

the direction of the track but a better estimate of the mass can be obtained in this way. The value of $p\beta$ obtained from 30 independent cells of 400 μm each is (455 ± 80) MeV/c at the decay point. A proton of $g^* = 2.34$ should have a $p\beta$ of 410 and a hyperon (Σ), 540 MeV/c. The mass is thus (1 020 ± 150) MeV. Again it is not really possible to decide between a hyperon and proton mass.

6. – The anti-proton star.

Having proved that the particle from the decay event is entering the interaction and not leaving, it is easy to show that the track cannot be caused by any known particle other than an anti-proton or anti-hyperon by examining the visible energy in the star. An attempt has been made to identify all the particles coming from the star. A list of the identities and the ranges is shown in Table II. The identity of the shower particles is especially important and

TABLE II.

Track No.	Identity	Range	Ionisation	Scattering measurements	Energy (MeV)
1	Proton or α-particle	256 μm	—	—	6
2	Proton	1.56 cm	—	—	67
3	$\pi^{\pm}$-meson	—	$g^* = 1.31 \cdot$ min at star	$p\beta$ at star = (210 ± 30) MeV/c	87
4	Proton (+)	96 μm	—	—	3
5	Proton (+)	576 μm	—	—	10
6	Proton (+)	156 μm	—	—	5
7	π^--meson	—	$g^* = 1.21 \cdot$ min at star, it interacts after 9.8 cm when $g^* = 2.7$	$p\beta$ at star = 190 MeV/c $p\beta$ at intern. = 70 MeV/c	108
8	Proton (+)	56 μm	—	—	2
9	α-particle	106 μm	—	—	15
10	π^+-meson	3.5 cm	$g^* = 1.70$ at star	$p\beta$ at star = 86 MeV/c	50
11	α-particle	46 μm	—	—	9
				Total visible energy	362
				Binding energy	70
				Neutron estimate	93
				Neutron B. E.	50
				$\pi^{\pm}$-meson rest mass	420
				π^0-meson K. E. estimate	122
				π^0-meson rest mass	210
				Total energy released	1327

(+) We cannot exclude the possibility that these tracks may be due to deuterons. However, their energies have been determined on the assumption that they are protons.

these tracks have had multiple scattering measurements made where possible together with ionization measurements. In one case the track was arrested in the emulsion and it produced a typical π-μ decay event.

Track No. 2 has been arrested after 1.56 cm and is a proton of 67 MeV.

Track No. 3 has an ionization of $1.31 \times$min ($\pm .05$). This track could not be traced through after the first six pellicles and so no information on the range is available. Scattering measurements however give a $p\beta$ of (210 ± 30) MeV/c from 40 independent cells of 100 μm. If the track were a proton it should have a $p\beta$ of 1800, a π-meson should have 140 MeV/c. It has therefore been taken as a π-meson of 87 MeV.

Track No. 7 has an ionization of $(1.21 \pm 0.03) \times$min and it interacts after 9.8 cm when its grain density is 2.7 ± 0.1. Thus by $\Delta g^*/\Delta R$ it is a π-meson. Confirmation from scattering gives $p\beta = 190 \pm 40$ from 25 independent cells of 100 μm at the anti-proton star and (70 ± 15) MeV/c from 25 cells at the interaction. As the energy at the point of interaction is only 16 MeV, it is probably of negative charge. The secondary prongs from the interaction are both protons of 55 and 5 MeV.

Track No. 10 has been arrested after 3.51 cm. It decays into a normal μ-meson and is therefore a π-meson of 50 MeV. Scattering confirms this, giving a $p\beta$ of 86 MeV/c with an ionization of $g^* = 1.7 \times$minimum.

The total visible energy as can be seen from the table is 363 MeV + 420 MeV in the rest masses of the shower tracks. The total is thus 783 MeV. If we include 8 MeV binding energy per nucleon and include $\frac{1}{2}$ the total $\pi^{\pm}$-meson energy for neutral π^0-mesons and an equal amount of evaporation energy for neutrons as for the protons, one obtains a total energy of 1327 MeV—a little lower than that available for an anti-proton annihilation — 2108 MeV.

With a total visible energy as high as 782 MeV all interactions involving hyperons, K-mesons or π-mesons are ruled out as explanations and the only tenable hypothesis is that it was caused by the annihilation of an anti-proton or anti-hyperon. The latter is virtually impossible, because of insufficient energy in the beam to create anti-Σ hyperons. They can of course be produced by a secondary interaction of anti-proton: $\bar{p} + p \rightarrow \bar{\Sigma}^+ + K^- + n$.

7. – The Q-value in the decay.

The best value of the mass of the Λ^0 has recently been obtained by the Barkas group (7), it is 1115.3 MeV and leads to a Q-value of 37.45 MeV. From the opening angle of the « V » and the energies of the secondary par-

(7) W. H. BARKAS: *Report to the Padua-Venice Conference*, (Sept. 1957).

ticles, the best Q-value obtained for this event is $(35^{+2.6}_{-0.9})$ MeV. The main contribution to the error comes from the energy of the anti-proton from the ionization measurement. Reducing the plateau minimum ratio in nuclear emulsion would have the effect of increasing the Q-value. The dependence of the Q-value on θ, the opening angle, and on $E_{\bar{p}}$ is shown in Figs. 4 and 5. The

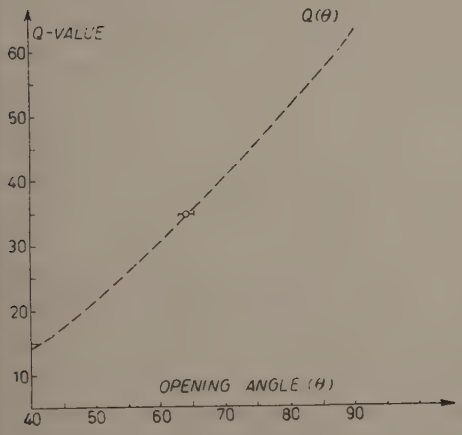

Fig. 4.

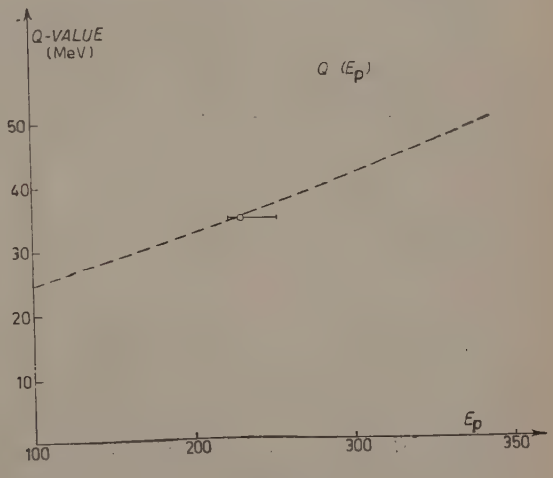

Fig. 5.

error in the opening angle $\pm 1^\circ$ contributes ± 0.8 MeV to the error in the Q-value, while the statistical error in $E_{\bar{p}}$ contributes ± 0.7 MeV. The Barkas range energy relation ([8]) has been used and the density of the emulsions at the time of exposure was determined by obtaining the mean range of flat μ-mesons from π-meson decays found during the course of the experiment.

8. – Conclusion and discussion.

The ionization measurements on the anti-proton track, the visible energy in the interaction and the apparent Q-value all point to the conclusion that the event represents the decay of an anti-lambda hyperon. The only tenable alternative, if one accepts the anti-proton direction, is that the decay event is an interaction produced by an anti-neutron. We can never exclude this possibility but it is surely very unlikely.

([8]) W. H. Barkas, P. H. Barrett, P. Cuer, H. H. Heckman, F. M. Smith and H. K. Ticho: *Nuovo Cimento*, **8**, 185 (1958).

If one does not accept the direction of the anti-proton track, one can postulate that the large interaction is a neutral star which gives rise to a track which either decays or interacts to produce a single π-meson of 32 MeV. The ionization measurements at the decay point and at the interaction enable one to exclude a number of decay possibilities:

a) $K^{+}_{\pi 2}$-decay in flight. The range of the π-meson defines the momentum of the K-meson at the decay point as 370 MeV/c. The grain density of the track should therefore be 2.1. This is within the error of the measured value but the grain density at the interaction is 8 standard deviations from that expected. The $p\beta$ measurement would also have to be wrong.

b) Σ^{+}-hyperon decay in flight. Again the momentum of the hyperon is defined and the grain density should be 2.56, 8 standard deviations from the measured value.

c) A positive cascade hyperon decay in flight. The momentum is defined as 460 MeV/c which would result in a grain density of 1.80, 6 standard deviations away from the measured value.

The interaction of a positive or negative Σ-hyperon to give a single positive π-meson is also very unlikely. Hence even if the direction of the anti-proton is not believed it is a very difficult event to explain away by an explanation involving known particles.

* * *

We are greatly indebted to Dr. E. J. Lofgren and his colleagues at the Berkeley Bevatron for making the exposure possible, and to Professor D. H. Wilkinson and Dr. Warren Chupp who carried out the exposure. Dr. C. Waller of Ilford Inc. is again to be thanked for the careful preparation of the stack. We thank Professor N. Dallaporta and Professor C. F. Powell, Dr. D. Evans, Dr. H. Huzita, Dr. S. Natali, Professor Gerson Goldhaber, Dr. Sulamith Goldhaber, Professor D. H. Stork and Professor H. K. Ticho for many fruitful discussions.

We appreciate the encouragement of Professor N. Dallaporta, Professor M. Merlin, Professor C. F. Powell and Professor J. R. Richardson. D. J. P. thanks Professor C. F. Powell for the loan of plates, Professor N. Dallaporta and the University of Padua for hospitality in the fall of 1957 and Professor J. R. Richardson for the kind hospitality of his laboratory during 1958.

Appendix

As an example of what we can expect kinematically we show in Fig. 6 two curves which give the maximum angle of emission in the laboratory system plotted against the energy of the Λ^0. In curve 1 we have assumed the momentum of the incoming π-mesons to be 5 GeV/c, the proton to be moving

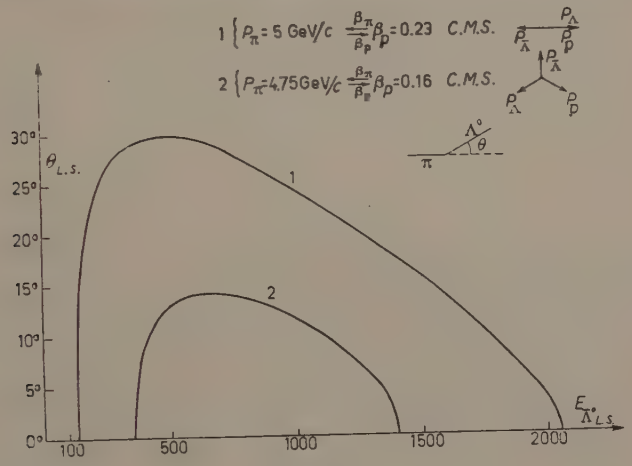

Fig. 6.

in the opposite direction to the π-meson with a velocity of $\beta = .23$, and that the momentum of the $\bar{\Lambda}^0$ hyperon in the c.m.s. is opposite to both the momenta of the Λ^0 and the proton.

All values of θ_{Lab} and $E_{\bar{\Lambda}^0}$ within the curve are permitted; curve 2 gives an example for an intermediate case.

RIASSUNTO

Un evento, trovato in un pacco di emulsioni nucleari esposte a 10^6 π^- di 4.6 GeV/cm^2 è stato interpretato come un esempio del decadimento in volo di un anti-iperone, $\bar{\Lambda}^0$. L'evento e l'interpretazione sono discussi in dettaglio.

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A Search for Anisotropy of Inertia.

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(ricevuto il 6 Agosto 1958)

Summary. — One of the possible consequences of the Mach principle is that the asymmetric position of the Galaxy with respect to the Earth induces an asymmetry in inertia. A test of this possibility is proposed. It consists in the microwave measurement of the Zeeman splitting of electron levels having different orientations relative to the direction Earth-Center of the Galaxy. The isotropy of inertia on the Earth can thus be checked with a precision of about 1 part in 10^{11}.

One of the fundamental assumptions of mechanics and gravitational theory is that the equality of the inertial and gravitational masses of a body is *exact*, and independent of the position and motion of the body. This assumption is an extrapolation to infinite accuracy of observations which obviously have a finite precision. It is also an observational fact (of finite precision) that the so-called « fixed stars » do in fact represent an inertial reference frame.

This last observation has led to the formulation of Mach's principle, which asserts that the inertia of a body at any point in the Universe is determined by « the fixed stars », which must be understood as the total distribution of matter in the Universe.

This distribution, however, is uniform and isotropic only to a certain degree of approximation. If Mach's principle holds, we might then expect that the slight asymmetries in the distribution of matter at large would result in slight deviations from at least some of the laws of mechanics and gravitation which are commonly assumed to be exact. It is thus desirable to find tests that can demonstrate the validity of Mach's principle and of its consequences. In this

article we shall concentrate on the question of the isotropy of inertia and describe a relevant experiment more sensitive than those performed thus far.

To good accuracy, the acceleration produced on any body by a force points in the same direction as the force, and the ratio of force to acceleration, the inertial mass, is independent of the direction of the force; *i.e.*, inertia is isotropic, and inertial mass is a scalar quantity. However, if Mach's principle holds, these statement should be true only in first approximation and asymmetries in the matter distribution at large, say concentration of matter near the center of our Galaxy, could produce asymmetries in inerta.

Mach's principle alone does not specify the nature of the effect that matter at large has on the inertia of a body; however, a rather plausible possibility for the contribution to the inertia of a test-body, resulting from a mass M a vector distance $\boldsymbol{r}$ away, is the following:

a) The contribution is proportional to M and to a negative power, $-\nu$, of the distance $\boldsymbol{r}$.

b) The contribution depends on the angle θ between $\boldsymbol{r}$ and the direction of acceleration of the test-body (1).

The most attractive possibility for this θ-dependence, which still preserves energy conservation in the usual sense is that the contribution to the inertia of the test-body has a maximum value for $|\cos\theta| = 1$ (motion parallel or antiparallel to $\boldsymbol{r}$) and is zero (2) for $\cos\theta = 0$ (motion in a plane perpendicular to $\boldsymbol{r}$ (3)).

With these assumptions, the inertial mass of a test body becomes a symmetric tensor $\mathcal{M}_{ij}$ for any distribution of matter in the Universe. *E.g.*, Newton's

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(1) One cannot, of course, exclude the other possibility that the contribution to inertia is independent of θ. In this case inertia would still be isotropic for any distribution of matter in the Universe, although the numerical value of the inertial mass of a test-body would depend on its position in the Universe.

(2) The more general case of a non-zero minimum for $\cos\theta = 0$ gives qualitatively similar results, except that the expressions for $\Delta\mathcal{M}/\mathcal{M}$, to be discussed later, are decreased by a constant factor.

(3) The assumption that inertia might be larger when the test body is, for instance, accelerated toward M and smaller when accelerated in the opposite direction, has the rather drastic consequence that ordinary forces would no longer be conservative in the usual sense. It would follow that, *e.g.*, all atoms, molecules and, consequently, also bulk matter would be electrically polarized with the electric dipole moments pointing toward the Galactic Center; another consequence would be that any compound system with internal forces would be repelled by matter by amounts proportional to the strength of the internal forces. It would presumably be too fanciful to seek any connection between this effect and the observed expansion of the Universe.

law is generalized to

$$F_i = \sum_{j=i}^{3} \mathcal{M}_{ij} a_j ,$$

where F and a are force and acceleration, respectively, and the kinetic energy T becomes $\frac{1}{2}\sum_{i,j}(\mathcal{M}^{-1})_{ij} p_i p_j$, where $\boldsymbol{p}$ is the linear momentum. The mass tensor is characterized by the directions of the three principal axes and the corresponding diagonal elements $\mathcal{M}_{ii}$.

We shall be mainly interested in the effect of the mass concentration near the center of our own Galaxy. In this case, neglecting the mass distribution in the Galactic Disc, the mass tensor of a test-body has the direction to the Galactic Center as one principal axis with diagonal element $\mathcal{M} + \Delta\mathcal{M}$, the diagonal elements for the other two axes being $\mathcal{M} - \frac{1}{2}\Delta\mathcal{M}$ each.

The order of magnitude of the expected anisotropy can be evaluated as follows: If ϱ is the average density of matter in the Universe, the isotropic part $\mathcal{M}$ of the inertia of a body on the Earth is proportional to

$$\text{Inertial mass} = \mathcal{M} \propto \int_0^R \frac{4\pi r^2 \varrho}{r^\nu}\,\mathrm{d}r = \frac{4\pi\varrho}{3-\nu} R^{(3-\nu)} ,$$

where R is the maximum distance from which matter still influences the inertia on the Earth. The anisotropic contribution $\Delta\mathcal{M}$ to the local inertia from a body of mass M at a distance r away is thus

$$\frac{\Delta\mathcal{M}}{\mathcal{M}} = \frac{M}{r^\nu}\,\frac{3-\nu}{4\pi\varrho R^{(3-\nu)}} . \tag{1}$$

To give a numerical example, assume that $\varrho = 10^{-28}$ g cm^{-3}, and that $R = cT = 10^{28}$ cm, the so-called radius of the Universe deduced from the Hubble constant. Then for $\nu = 1$ [4] the extent of the anisotropy due to the matter near the center of the Galaxy would be of the order of 1 part in 10^7 and for $\nu = 0.25$ 1 part in 10^{11}.

It must be emphasized here that these numbers depend strongly on the values assumed for the density and the radius of the Universe, quantities which are not yet known with any accuracy. (The corresponding quantities for the Galaxy are instead rather sound.) The value of R is particularly uncertain since the value of the Hubble constant is derived only by extrapolation

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[4] Values of ν larger than 1 can be excluded *a priori* since they would make the masses of the Sun and of the Earth prominent.

of the red shift from relatively small distances and the detailed relation between optical red shift and decrease of inertial influence of matter is not known. In fact, it is likely that our assumed value of R is an underestimate.

In principle, this anisotropy would be observable in classical macroscopic experiments; *e.g.*, there should be a diurnal variation in the period of a quartz crystal (or a pendulum) clock if one assumes that the anisotropy of electric forces (or gravitational mass) is either absent or at least different from that of the inertial mass. However, present-day, observations place an upper limit of about 1 part in 10^8 on a possible anisotropy observable with a quartz clock.

We want to propose a more sensitive test that utilizes microwave measurements of the splitting of atomic energy levels. In principle it consists in the comparison of the periodic motion ot two atomic electrons, one moving in a line pointing towards our Galactic Center, the other moving in a plane perpendicular to this direction. The comparison is realized by measuring the energy difference of the two levels.

Consider, for example, a single-electron atom in a static magnetic field with orbital quantum number l and magnetic quantum number m_l referred to the field direction as axis (component of orbital angular momentum parallel to this axis). For $m_l = 0$ (or small) the electron moves preponderately parallel to this axis; for $|m_l|$ equal (or close to) l there is preponderance of motion in a plane perpendicular to this axis.

What then must be measured is either the fine structure splitting between two single-electron levels with the same value of m and differing values of j, or the Zeeman splitting between two levels with the same value of j and different values of m. What one needs is only the *variation* of such a fine structure or Zeeman splitting as the orientation of the magnetic field (which determines the m-axis) relative to the Galactic Center is changed. This should be observed either by rotating the experimental apparatus or merely by looking for sidereal variations (as the orientation of the laboratory co-ordinates relative to the Galactic Center is modified by the Earth's rotation).

For a given value of $\Delta\mathcal{M}/\mathcal{M}$, estimated in eq. (1), the energy shift due to any anisotropy of inertia is given by

$$\Delta E = \frac{\Delta\mathcal{M}}{\mathcal{M}} \bar{T} \bar{P}_2 , \tag{2}$$

where $\bar{T}$ is the average kinetic energy of the single (or valence) electron and $\bar{P}_2$ is the expectation value of the Legendre polynomial of order 2 with the direction to the Galactic Center as polar axis. This perturbation is equivalent to that of a weak quadrupole field and $\bar{P}_2$ changes sign when the magnetic field direction (with quantum numbers fixed) changes from parallel to perpendicular to the direction of the Galactic Center.

We quote a few examples for $\bar{P}_2$ with the magnetic field pointing to the Galactic Center and for a single p-electron. Let l, j, I and F be the orbital, total electronic, nuclear spin and total angular momentum (electronic plus nuclear spin) quantum numbers; m_l, m_j, m_I and m_F the corresponding magnetic quantum numbers. In the absence of hyperfine structure ($I = 0$) $\bar{P}_2$ is zero for $j = \frac{1}{2}$; $\bar{P}_2 = -\frac{1}{5}$ for $m_j = \pm\frac{3}{2}$, $\bar{P}_2 = +\frac{1}{5}$ for $m_j = \pm\frac{1}{2}$ if $j = \frac{3}{2}$ ($p\,\frac{3}{2}$-electron). For non-zero nuclear spin I and very weak magnetic field (Zeeman effect $\ll$ hyperfine structure) and $F = I + j$ we have $\bar{P}_2 = -\frac{1}{5}$ for $m_F = F$ and $\bar{P}_2 = (j - I)/5(j + I)$ for $m_F = F - 1$.

Precision experiments of the kind described above have of course been performed, although they were not designed for investigating anisotropy of inertia. One example is the microwave experiments of LAMB and collaborators on fine structure splitting in hydrogen and deuterium ([5]), which measure the energy differences between $2p_{\frac{1}{2}}$ and $2p_{\frac{3}{2}}$-states (about 10^4 MHz while the average energy corresponds to about $3 \cdot 10^9$ MHz).

Still more sensitive tests for anisotropy of inertia are provided by precision microwave measurements of Zeeman transitions in very weak magnetic fields for atoms with ground states with $j > \frac{3}{2}$. Relevant experiments which have already been carried out are those of KUSCH and collaborators ([6]) on Zeeman transitions in the $p_{\frac{3}{2}}$-states of Ga and In.

However, since the measurements conducted thus far were not intended for observing sidereal variations, the various hours in the sidereal day where not covered in a systematic manner, and the orientations of the apparatus (relative to the polar axis) were not chosen to give the maximum sidereal variation ([7]). Nevertheless sidereal variations in the Zeeman splitting energies as large as 0.2 MHz, say, would probably have been noticed by the experimenters and the negative results so far place an upper limit of about 10^{-9} on $\Delta\mathcal{M}/\mathcal{M}$.

Further improvement in sensitivity could easily be achieved by similar experiments repeated with the purpose of studying sidereal variations or dependence on the orientation of the apparatus relative to the Galactic Center. Very weak magnetic fields giving Zeeman splitting energies less than 100 MHz, say, transition between $F = I + j$, $m_F = F$ and $F = I + j$, $m_F = F - 1$ in atoms with relatively small nuclear spins and $p_{\frac{3}{2}}$-ground states would be particularly favorable. In this manner sidereal variations of the order of one kHz might

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([5]) E. S. DAYHOFF, S. TRIEBWASSER and W. E. LAMB: *Phys. Rev.*, **89**, 106 (1953).

([6]) P. KUSCH and H. M. FOLEY: *Phys. Rev.*, **74**, 250 (1948); A. K. MANN and P. KUSCH: *Phys. Rev.*, **77**, 435 (1950).

([7]) At latitudes near 45° N the most favorable direction is the NS direction in a horizontal plane. This line points towards the center of our Galaxy (due S) at sidereal time about 17.30 h; twelve hours later the Galactic Center is at the nadir.

be detectable and one might hope to reach accuracies for $\Delta\mathcal{M}/\mathcal{M}$ of the order of one part in 10^{11} or more. With such accuracy any anisotropy due to the presence of our Galaxy would be detectable if the parameter ν in eq. (1) is around unity, even if the value of R is substantially larger than that used in the example discussed earlier.

* * *

The authors are grateful to Professor W. LAMB for helpful suggestions; they also wish to thank the Gravity Research Foundation, New Boston, New Hampshire, for the interest shown in the ideas expressed in this article.

RIASSUNTO (*)

Una delle possibili conseguenze del principio di Mach è che la posizione asimmetrica della Galassia rispetto alla Terra induca un'asimmetria nell'inerzia. Si propone una verifica di tale possibilità, consistente nella misura delle microonde della separazione per effetto Zeeman dei livelli elettronici aventi differenti orientazioni rispetto alla direzione Terra-centro della Galassia. È così possibile verificare l'isotropia dell'inerzia sulla terra con una precisione di circa 1 a 10^{11}.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. X, N. 4 16 Novembre 1958

Decays of Hyperfragments in Nuclear Emulsion.

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(ricevuto l'8 Agosto 1958)

Summary. — Seven decays of hyperfragments have been found in Ilford G-5 emulsion. Five events are uniquely defined decays. Within the limits of error the binding energies agree with the results of other authors.

170 double stars in nuclear emulsion were selected for decays of hyperfragments. We used the criteria described by Filipkowski, Gierula and Zielinski in [1] for selection. Four decays have been found in plates which are part of the I-stack, three decays in plates exposed by the K^--beam of the Berkeley bevatron.

Table I.

Event no.	Primary star		Secondary star		Time of flight of the fragment (10^{-12} s)
	Produced by	Type	Kinetic energy of all charged particles (MeV)	Residual momentum (MeV/c)	
1	Cosm. rays	6+2n	34.6	—	7.5 (if $^5He_\Lambda$)
2	»	13+1n	33.1	30	4.7
3	»	K^--capt.	35.1	10	1.1
4	»	17+0n	68.9	400	2.3
5	K^--beam	K^--capt.	35.9	6	8
6	»	»	65.8	405	1.2
7	»	»	38.1	9	2 (if $^3H_\Lambda$)

[1] A. Filipkowski, J. Gierula and P. Zielinski: *Acta Phys. Polon.*, **16**, 139 (1957).

TABLE II. – *Range R, polar angle φ, dip angle ϑ, kinetic energy T and relative grain density g/g_0 of the secondary star tracks.*

Event no.	Track	R (μm)	φ (*) (deg.)	ϑ (*) (deg.)	g/g_0	Ident.	T (MeV)
1	F	125 ± 6	348 ±2	+66±3		$^{4,5,7}He_\Lambda$	18 ±1 (if $^5He_\Lambda$)
	1	31.4± 0.6	121.5±1	−25±3		^1H	1.7 ±0.1
	2	≈ 2	≈ 120	≈+45		$^{3,4,6}He$	0.63±0.05
	3	17515 ± 50	301.5±1	−16±3	2.0 ±0.15	π^-	32.25±0.7
2	F	68 ± 2	9 ±1	+27±3		$^5He_\Lambda$	12 ±0.5
	1	14.3± 2	32 ±6	≈+40		^4He	3.92±0.52
	2	277 ± 12	177 ±3	−77±3		^1H	6.56±0.2
	3	9580 ±100	216 ±1	−26±3	2.5 ±0.25	π^-	22.61±0.54
3	F	11.4± 0.5	316 ±5	−30±5		$^5He_\Lambda$	3.6 ±0.2
	1	10.4± 0.5	118.5±3	0±5		^4He	3.1 ±0.23
	2	10490 ± 50	247.5±1	−23±2	2.55±0.1	π^-	23.85±0.54
	3	400 ± 4	327.5±1	+20±2		^1H	8.18±0.11
4	F	32 ± 1	66 ±2	0±5		$^7Li_\Lambda$	13 ±0.5
	1	> 8500	8.5±1	−39±2	4.4 ±0.45	^1H	59.3 ±2.8
	2	35.5± 6.5	224 ±2	+20±5		^3H	2.6 ±0.34
	3	196 ± 5	303 ±1	+47±2		^2H	6.97±0.15
5	F	76 ± 1	350 ±3	+17±5		$^4H_\Lambda$	4.8 ±0.2
	1	85.5± 2	359 ±2	+39±3		^3H	4.8 ±0.1
	2	214 ± 4	143.5±1	−37±2		^1H	5.62±0.1
	3	11760 ±140	214 ±2	−31±3	2.1 ±0.15	π^-	25.50±0.61
6	F	15 ± 1	104 ±2	−28±5		$^7Li_\Lambda$	7.2 ±1
	1	9635 ±170	26 ±1	+42±2	4.5 ±0.2	^1H	51.1 ±0.75
	2	53.5± 1	43 ±2	−35±5		^3H	3.47±0.08
	3	423 ± 6	273 ±1	+43±2		^2H	11.16±0.16
7	F	320 ± 4	319.5±1	+10±3		$^{2,3,4}H_\Lambda$	11.3 ±0.3 (if $^3H_\Lambda$)
	1	282 ± 3	140.5±1	−25±2		^1H	6.63±0.1
	2	4.2± 0.5	≈ 255	≈ −10		$^{1,2,3}H$	0.45±0.1
	3	16360 ±120	342.5±1	+31±2	1.8 ±0.1	π^-	31.00±0.71

(*) φ is the angle between the track and an arbitrarily chosen direction, both projected in the plane of the emulsion. ϑ is the angle between the track and the plane of the unshrinked emulsion.

For the identification of singly charged particles we have used the range-energy-relation of BARKAS (2), for multiply charged particles the curves of VIGNERON (3) and PAPINEAU (4). In computing binding energies the tables of SEGRÈ (5) (isotopic masses) and ROSENFELD (6) (masses of elementary particles) have been employed. The following errors have been taken into account: *a*) Uncertainty of range-energy-relation ($\pm 0.60\%$, s. (7)); *b*) Error of range measurement; *c*) Variation of stopping power ($\pm 0.5\%$, s. (8)); *d*) Heterogeneity of emulsion (s. (9)); *e*) Range straggling (s. (9)); *f*) Uncertainty in Q_Λ (s. (6)).

In all events reported here we have assumed that the hyperfragment decayed after coming to rest in the emulsion. The lower limit of the life time of the hyperfragments following from their time of flight is then 10^{-12} s.

The interesting data of the events have been summarized in the Tables I and II. A drawing of each event is shown in Figs. 1-7 (*A*, primary star; *B*, secondary star; *F*, interconnecting track).

Event No. 1. – The tracks of the secondary star are coplanar within the limits of error. The interconnecting track *F* has no visible δ-rays or gaps. A few μ before producing star *B* the fragment undergoes large-angle Coulomb scattering. Therefore it is supposed that the decay occurred at rest. Particle 3 is a π^--meson producing a σ-star at the end of its range. Track 1 has four visible gaps suggesting a charge $z \leqslant 2$. Assuming that particle 1 is a proton and particle 2 a helium nucleus, the total momentum will be zero. The decay scheme will therefore appear as follows:

Fig. 1.

$$^{4,5,7}\mathrm{He}_\Lambda \rightarrow {}^{1}\mathrm{H} + {}^{3,4,6}\mathrm{He} + \pi^- + Q\,.$$

Event No. 2. – The tracks of the secondary star are coplanar within the limits of error. Track *F* has no visible δ-rays and undergoes wide angled

(2) W. H. BARKAS: private communication (March 1957).
(3) L. VIGNERON: *Journ. Phys. et Rad.*, **14**, 145 (1953).
(4) A. PAPINEAU: Rapport C.E.A. no. 543 (1956).
(5) E. SEGRÈ: *Experimental Nuclear Physics*, vol. **1** (1953).
(6) A. H. ROSENFELD: private communication (April 1957).
(7) Genoa-Milan Report no. 5.
(8) R. LEVI SETTI, W. E. SLATER and V. L. TELEGDI: preprint April 1957.
(9) W. H. BARKAS, F. M. SMITH and W. BIRNBAUM: *Phys. Rev.*, **98**, 605 (1955).

Coulomb scattering before producing star B. The end of track 3 seems to be a typical $\varrho\pi$-event. The interpretation of this track as π^--meson has been supported by grain counting. Track 1 and 2 have been identified by momentum balance. The ionization of both tracks suggests a charge of $z \leqslant 2$. If track 1 is interpreted as an α-particle and track 2 as a proton, the residual momentum is 30 MeV/c corresponding to a neutron of 0.5 MeV energy. If we consider the emission of such a low-energetic neutron as improbable and the residual momentum as due to the error of measurement, we can only take one decay scheme into account:

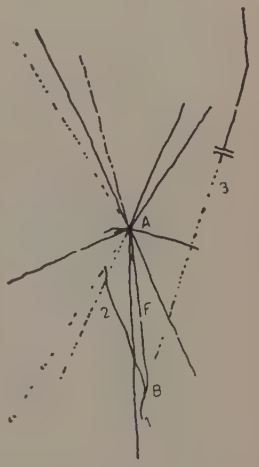

Fig. 2.

$$^5\mathrm{He}_\Lambda \rightarrow {}^4\mathrm{He} + {}^1\mathrm{H} + \pi^- + Q\,.$$

For the binding energy B_Λ we have found

$$B_\Lambda = (3.8 \pm 0.8)\ \mathrm{MeV}\,.$$

Event No. 3. – The tracks of the secondary star are coplanar within ± 4 degrees. Particle 2 is a π^--meson producing a σ-star at the end of its range. Track 3 could be identified as a singly charged particle by measurement of the total gap length. If it is interpreted as a proton and particle 1 as an α-particle, the residual momentum is only 10 MeV/c. The decay process therefore must be as follows:

Fig. 3.

$$^5\mathrm{He}_\Lambda \rightarrow {}^4\mathrm{He} + \pi^- + {}^1\mathrm{H} + Q\,;$$

$$B_\Lambda = (1.8 \pm 0.6)\ \mathrm{MeV}\,.$$

Event No. 4. – The tracks of the secondary star are not coplanar. A remarkable thinning-down of the fragment track suggests a charge of $z \geqslant 3$. Track 3 leaves the emulsoin. The particle has been identified by measuring the grain density at different places of the track. The ionization of track 2 and 3 suggests a charge of $z \leqslant 2$. Assuming this we computed all possible decay schemes. When the emission

of more than one neutron is excluded, only one reaction is in agreement with the hyperfragment interpretation, namely:

$$^{7}\mathrm{Li}_{\Lambda} \rightarrow {}^{1}\mathrm{H} + {}^{3}\mathrm{H} + {}^{2}\mathrm{H} + \mathrm{n} + Q\ ; \qquad B_{\Lambda} = (3.7 \pm 5.2)\ \mathrm{MeV}\ .$$

Fig. 4.

Fig. 5.

Event No. 5. – The tracks of the secondary star are coplanar within ± 2 degrees. The fragment undergoes wide angled Coulomb scattering a few μ before producing star *B*. The end of track 3 seems to be a $\varrho\pi$-event. This assumption has been supported by measurement of grain density. Assuming that particle 1 is a triton and particle 2 a proton, the total momentum will be zero. This interpretation agrees with the appearance of the tracks. Only one decay scheme can therefore be taken into account:

$$^{4}\mathrm{H}_{\Lambda} \rightarrow {}^{3}\mathrm{H} + {}^{1}\mathrm{H} + \pi^{-} + Q\ ; \qquad B_{\Lambda} = (1.0 \pm 0.7)\ \mathrm{MeV}\ .$$

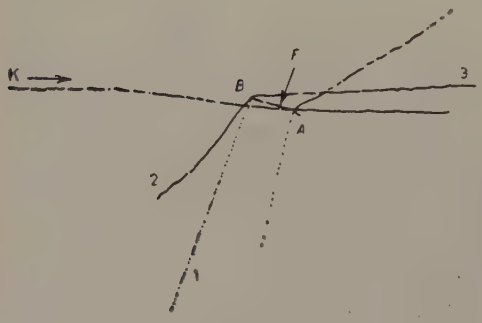

Fig. 6.

Event No. 6. – The tracks of the secondary star are not coplanar. From an energetic analysis of the primary star follows that, beyond reasonable doubt, the secondary star has been produced neither by a collision of a stable particle nor by the capture of a negative meson or hyperon. Track 1 has been identified as proton track by grain density. The ionization of tracks 2 and 3 suggests

a charge of $z \leqslant 2$. Assuming this we considered all possible decay schemes with the emission of only one neutron Only one case agrees with the hyperfragment interpretation, namely:

$$^{7}\mathrm{Li}_{\Lambda} \rightarrow {}^{1}\mathrm{H} + {}^{3}\mathrm{H} + {}^{2}\mathrm{H} + \mathrm{n} + Q \; ;$$

the binding energy is

$$B_{\Lambda} = (5.1 \pm 1.6) \text{ MeV} .$$

The decay is uniquely defined, if the emission of two or more neutrons is excluded.

Event No. 7. – The tracks of the secondary star are coplanar within the limits of error. Tracks F and 1 have been identified as singly charged particles by measurement of the total gap length. Particle 3 is a π^{-}-meson pro-

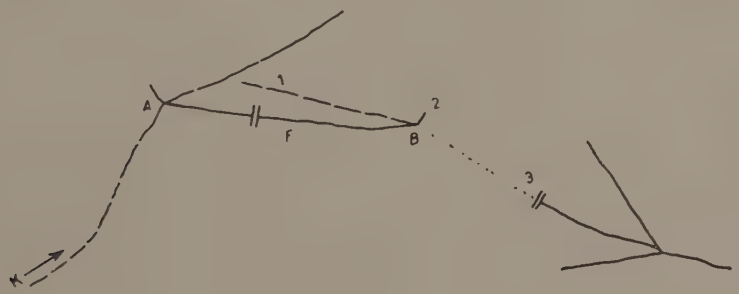

Fig. 7.

ducing a σ-star at the end of its range. From momentum balance follows that track 3 corresponds to a hydrogen nucleus and track 1 to a proton. The decay process must therefore be as follows:

$$^{2,3,4}\mathrm{H}_{\Lambda} \rightarrow {}^{1}\mathrm{H} + {}^{1,2,3}\mathrm{H} + \pi^{-} + Q \; ; \qquad Q \approx 38 \text{ MeV} .$$

For the binding energy we find

$$B_{\Lambda} = (-1.1 \pm 0.8) \text{ MeV} .$$

* * *

I am greatly indebted to Dr. K. LANIUS for his encouragement and for the opportunity of many critical discussions. I further wish to thank Mr. D.

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RIASSUNTO (*)

Si sono trovati sette decadimenti di iperframmenti in emulsione Ilford G-5. Cinque eventi sono decadimenti univocamente accertati. Nei limiti d'errore, le energie di legame si accordano coi risultati di altri autori.

(*) *Traduzione a cura della Redazione.*

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An Investigation on Plasmas in External Magnetic Fields (*).

II. - Varying Fields.

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(ricevuto l'8 Agosto 1958)

Summary. — The behaviour of a plasma placed in a varying external magnetic field is investigated considering two limiting cases: 1) the ideal hydromagnetic ($\sigma \to \infty$); 2) the collisionless one. Restricting the calculations to general cylindrical geometries ($\partial/\partial z = 0$) and quasi-stationary processes, a complete set of plasma equations are obtained for both cases. These include among others equations of state also for the tenuous plasma. There is a close resemblance between the two sets of equations referring to completely different physical conditions. The equations can be readily solved to express every macroscopic plasma quantity in terms of the external magnetic field strength. The extension of our considerations to non-quasi-stationary processes is briefly depicted. It leads to a phenomenon which suggests new ways for the increase of plasma temperatures.

1. – Introduction.

In paper I (1) the stationary behaviour of a quasi-neutral plasma placed in an external magnetic field was investigated. With the exception of the deri-

(*) The present work was essentially finished by the end of 1957 and a full account of the basic ideas was given on the Ninth Meeting of the Israel Physical Society, April 2, 1958. (*Bulletin of the Research Council of Israel*, Section F, vol. 7F, p. 99). Due to the sharp expansion of literature on the subject several papers as refs. (2-4) have appeared during the last year covering some parts of the present work.

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(1) G. SCHMIDT: *Nuovo Cimento*, **10**, 55 (1958).

(2) E. N. PARKER: *Phys. Rev.*, **107**, 924 (1957).

(3) IA. P. TERLETSKIJ: *Journ. Exp. Theor. Phys.*, **5**, 755 (1957).

(4) *Nucleonics*, **16**, no. 5, 122 (1958).

vation of the pressure balance equation—which has been performed in full generality—certain restricting assumptions were made:

1) A tenuous plasma was considered neglecting collisions among particles.

2) Only « 2-dimensional geometries » were treated (straight and parallel $\boldsymbol{B}$ lines).

In the present paper plasmas in time dependent fields are investigated. It is generally assumed here that the variation of the field is slow, therefore the process can be treated in a quasi-stationary way, *i.e.* the system is moving through equilibrium states as described in paper I. The condition of 2-dimensional geometries is also retained.

The physical problem to be investigated is the following: A plasma of finite dimensions in the x-y plane, is placed in an initially homogeneous magnetic field $\boldsymbol{B}$ $(0, 0, B_z)$. Owing to plasma diamagnetism this field becomes distorted, so that in the steady state the pressure balance equation is satisfied. If now the external field is gradudally changed, the plasma characteristics must also change. It is shown that increasing the field, the plasma becomes gradually compressed and its energy (or kinetic temperature) raised. On the other hand, on decrease of the field, plasma expansion and cooling occur. Owing to the quasi-stationary condition these two processes take place in a symmetrical way.

Two opposite limiting cases are considered. The first one is approximately valid for very dense plasmas, with frequent particle-particle interaction. In this case the plasma behaves similar to a fluid with very high conductivity, and therefore the magneto-hydrodynamic equations can be applied. For the sake of simplicity the conductivity is assumed to be infinite.

The second case is that of tenuous plasmas as considered in paper I. In this article, however, a further restriction is also made; the particle motion is treated in the framework of the usual « first order theory ». It is hoped that a more general treatment on the lines of paper I will be forthcoming later.

The calculation of these two extreme cases leads to very similar results. Validity of the assumptions made, and order of magnitudes of the parameters for realizing these processes under laboratory circumstances are estimated in Appendix A.

Appendix B deals with considerations for non-quasi-stationary systems, where an interesting new phenomenon occurs. Calculations for a simplified model show that compression and expansion are no longer symmetrical processes, and a periodic operation leads to continually growing kinetic temperatures.

2. – Magneto-hydrodynamic approximation.

We investigate the case of a fluid, with very high electrical conductivity σ placed in a varying magnetic field. The variation of the magnetic field inside the fluid is governed by the familiar equation expressing the « constancy of magnetic flux ». This can be obtained as follows:

The current density $\boldsymbol{i}$, in a fluid element moving with the velocity $\boldsymbol{v}$, in the local magnetic field $\boldsymbol{B}$, and electric field $\boldsymbol{E}$, equals:

$$\boldsymbol{i} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) . \tag{1}$$

Taking the curl of eq. (1) and making use of Maxwell's equations $\operatorname{curl} \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$, we obtain:

$$\operatorname{curl} \boldsymbol{i} = \sigma \left(- \frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{curl} \boldsymbol{v} \times \boldsymbol{B} \right) . \tag{2}$$

With the help of the equation $\operatorname{curl} \boldsymbol{H} = \boldsymbol{j}$ where the displacement current is neglected, as σ is high and $\partial/\partial t$ small, it follows:

$$- \frac{1}{\mu\sigma} \nabla^2 \boldsymbol{B} = \left(- \frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{curl} \boldsymbol{v} \times \boldsymbol{B} \right) , \tag{3}$$

where use has been made of $\operatorname{div} \boldsymbol{B} = 0$. Integration of eq. (3) over a « material surface » $\boldsymbol{S}$, whose boundary $\boldsymbol{s}$ is fixed to the moving fluid elements, yields:

$$\int_{\boldsymbol{S}} \frac{1}{\mu\sigma} \nabla^2 \boldsymbol{B} \, \mathrm{d}\boldsymbol{S} = \int_{\boldsymbol{S}} \left[\frac{\partial \boldsymbol{B}}{\partial t} - \operatorname{curl} (\boldsymbol{v} \times \boldsymbol{B}) \right] \mathrm{d}\boldsymbol{S} . \tag{4}$$

The second term on the right hand side equals:

$$- \oint (\boldsymbol{v} \times \boldsymbol{B}) \, \mathrm{d}\boldsymbol{s} = \oint \boldsymbol{B} (\boldsymbol{v} \times \mathrm{d}\boldsymbol{s}) . \tag{5}$$

This is the time rate of flux change within the material surface owing to the motion of the boundary, while the first term on the right hand side of eq. (4) signifies the change of flux caused by the time variation of the field. The total time rate of change of the flux within the moving boundary is finally:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \int \frac{1}{\mu\sigma} \nabla^2 \boldsymbol{B} \, \mathrm{d}\boldsymbol{S} . \tag{6}$$

If $\sigma \to \infty$ this reduces to:

$$\frac{d\Phi}{dt} = 0 . \tag{7}$$

The connection between the magnetic field and the characteristics of the fluid is given by the equation of pressure balance. In case of the assumed two dimensional geometry ($\partial/\partial z = 0$) and for a quasi-stationary process this can be written in the form [1]:

$$\operatorname{grad}\left(\frac{B^2}{2\mu} + p\right) = 0 . \tag{8}$$

To connect the material pressure with other quantities characterizing the fluid an equation of state is needed. For example if the process is adiabatic:

$$V = V_0\left(\frac{p_0}{p}\right)^{1/\varkappa} , \tag{9}$$

where V is the plasma volume, p_0 and V_0 are initial values of pressure and volume, respectively. $\varkappa$ is closely connected with f, the number of degrees of freedom in the plasma:

$$\varkappa = \frac{2+f}{f} . \tag{10}$$

The expression for the variation of temperature with pressure is correspondingly:

$$T = T_0\left(\frac{p}{p_0}\right)^{(\varkappa-1)/\varkappa} . \tag{11}$$

Equations (7), (8), (9) and (11) give a complete description of the processes taking place in the plasma when the external magnetic field is varied.

Let us apply these equations to the case where the plasma is initially confined within a material container and no magnetic field is present at $t = 0$. Due to eq. (7) the magnetic flux within any closed material curve remains constant, in this case $\varphi = 0$. Consequently the magnetic field inside the plasma remains $B_{in} = 0$, while outside the fluid it is $B_{ext} = B(t)$. Along a line crossing the boundary the magnetic field is described by a step function.

The « step » at the fluid boundary means that surface currents have been induced in order to prevent the penetration of the external field. These sur-

face currents interacting with the external fields produce a force density $\boldsymbol{f} = \boldsymbol{i} \times \boldsymbol{B}$ directed into the fluid tending to compress it (Fig. 1). This is expressed by eq. (8) which yields after integration:

$$\left[\frac{B^2}{2\mu} + p\right]_{\xi_1}^{\xi_2} = 0 \,. \tag{12}$$

Choosing the point ξ_1 inside and ξ_2 outside the fluid, and observing that $B(\xi_1) = B_{\text{in}} = 0$ and $p(\xi_2) = 0$, eq. (12) becomes:

$$\frac{B_{\text{ext}}^2}{2\mu} = p_{\text{in}} \,. \tag{13}$$

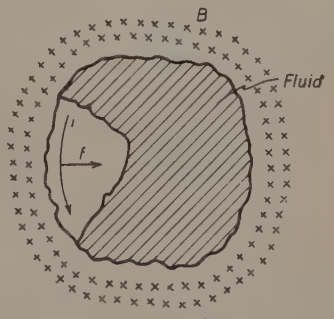

Fig. 1. – Conducting compressible fluid in a growing magnetic field.

As soon as $B^2(t)/2\mu \geqslant p_0$ the fluid detaches itself from the material walls and is confined merely by the magnetic pressure. Since $B_{\text{ext}}(t)$ is known, $p_{\text{in}}(t)$, $V(t)$ and $T(t)$ are also known functions through eqs. (13), (9) and (11). As $B(t)$ increases the pressure and temperature of the fluid increase as well, while the volume decreases according to the laws of an adiabatic compression, where the magnetic field assumes the role of a piston.

3. – Independent particle model.

When the plasma is so rare that particle collisions can be neglected, an entirely different physical picture must be adopted. Each particle moves in the electromagnetic field produced by external sources and by moving plasma particles. We therefore investigate the motion of a plasma particle under these conditions. Contrary to paper I where the stationary state of such a plasma was treated in general, we shall here restrict ourselves to the « first order theory », considering nearly circular particle paths, with the corresponding cyclotron radius superposed by a small drift.

The drift velocity is composed of different parts. In the stationary case every particle drifts along the lines $|\boldsymbol{B}| = \text{const}$ with the velocity:

$$\boldsymbol{w}_M = \frac{\frac{1}{2} m v^2}{q B^4} (\boldsymbol{B} \times \text{grad}\, B^2) \,, \tag{14}$$

where the index M denotes that this drift is due to the (inhomogeneous) magnetic field. The value of $\boldsymbol{B}$ is to be taken-according to the first order theory, at the guiding center.

When the external magnetic field is to be varied, the magnetic field inside the plasma undergoes a change and corresponding electric fields are induced. The equation of motion of a particle is:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\,. \tag{15}$$

The electric field is « transformed away » in a co-ordinate system moving with the velocity of « electric drift »:

$$\boldsymbol{w}_E = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}\,. \tag{16}$$

Indeed, putting

$$\boldsymbol{v} = \boldsymbol{w}_E + \boldsymbol{u}\,, \tag{17}$$

eq. (15) becomes:

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \frac{q}{m}(\boldsymbol{u}\times\boldsymbol{B}) - \frac{q}{m}\frac{\mathrm{d}\boldsymbol{w}_E}{\mathrm{d}t}\,, \tag{18}$$

by expanding the vector product $(\boldsymbol{E}\times\boldsymbol{B})\times\boldsymbol{B}$ and making use of $(\boldsymbol{E}\boldsymbol{B}) = 0$.

Expressing $\boldsymbol{w}_E$ as a series expansion in the neighbourhood of the guiding center eq. (16) yields:

$$\boldsymbol{w}_E = \left(\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}\right)_0 + (\boldsymbol{R}\,\mathrm{grad}_0)\boldsymbol{w}_E\,, \tag{19}$$

where the index 0 refers to the guiding center, and $\boldsymbol{R}$ has the magnitude of the cyclotron radius. Due to the assumption of small field variations in the range of a particle orbit the second term is small compared to the first. The quasi-stationary treatment demands on the other hand that $|\boldsymbol{w}_E| \ll |\boldsymbol{v}|$. The first term on the right hand side in eq. (19) is therefore of the first order while the second term is of second order and can be neglected in our approximation.

Eq. (18) has the following meaning: In a co-ordinate system moving with the velocity $\boldsymbol{w}_E = (\boldsymbol{w}_E)_0$ the particle moves with a velocity $\boldsymbol{u}$, affected only by the magnetic field and the inertial forces due to the acceleration of our system of reference. In the moving system $-\mathrm{d}\boldsymbol{w}_E/\mathrm{d}t$ is equivalent to a gravtational acceleration $\boldsymbol{g}$ and gives rise to the drift velocity ([5]):

$$\boldsymbol{w}_g = \frac{m}{q}\frac{\boldsymbol{g}\times\boldsymbol{B}}{B^2} = -\frac{m}{q}\frac{1}{B^2}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}\right)\times\boldsymbol{B}\right], \tag{20}$$

([5]) L. SPITZER: *Physics of Fully Ionised Gases* (New York, 1956).

with $\boldsymbol{E}$ and $\boldsymbol{B}$ taken again at the guiding center. In the quasi-stationary treatment of slow time variations $\boldsymbol{w}_g$ is a second order term.

It can be shown that, with suitably chosen « material surfaces » eq. (7) is again valid. Attaching the « material curves » to the guiding centers of particles, corresponding at a given moment to the same $|\boldsymbol{B}| = \text{const}$ lines (Fig. 2), the rate of change of flux is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{B}\,\mathrm{d}\boldsymbol{S} = \frac{\partial}{\partial t}\int_S \boldsymbol{B}\,\mathrm{d}\boldsymbol{S} + \oint \boldsymbol{B}(\boldsymbol{w}_M \times \mathrm{d}\boldsymbol{s}) + \oint \boldsymbol{B}(\boldsymbol{w}_E \times \mathrm{d}\boldsymbol{s})\,, \tag{21}$$

where second order terms are negelcted. In consequence of our choice of material curves $\boldsymbol{w}_M \times \boldsymbol{ds} = 0$. The first term on the right hand side can be transformed into a line integral with the help of a Maxwell equation; further substituting eq. (16) into the last term and expanding the double vector product, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi = -\oint \boldsymbol{E}\,\mathrm{d}\boldsymbol{s} + \oint \boldsymbol{E}\,\mathrm{d}\boldsymbol{s}\,, \tag{22}$$

where we made use again of $(\boldsymbol{EB})=0$. Eq. (22) shows that the magnetic field between any two $|\boldsymbol{B}| = \text{const}$ lines is frozen into this region. Apart from this restriction regarding regions for which eq. (22) is valid, the analogy with eq. (7) is apparent.

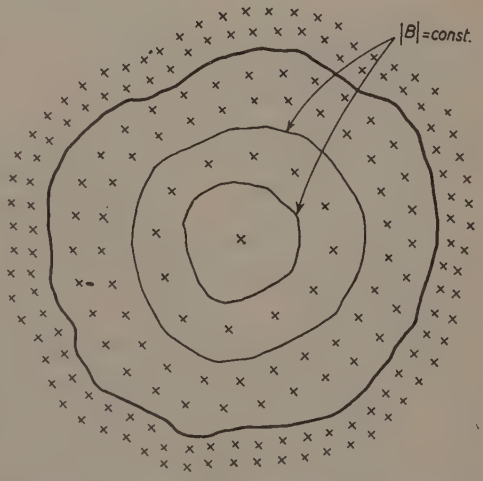

Fig. 2. – Tenuous plasma in a varying magnetic field.

As it was shown in paper I, the pressure balance equation is valid for any stationary plasma irrespective of collision rates. Eq. (8) can be used therefore without restrictions also in this treatment of non-colliding particles.

To complete the set of equations governing the plasma in varying fields, an equation of state is again needed. This can be obtained, by further investigation of the particle motion.

Let us assume for example that the external magnetic field is increasing. To satisfy condition (22), the regions S must shrink and the plasma becomes compressed. As the flux crossing decreasing material surfaces tends to be constant , B is increasing even in the co-ordinate system moving with the

guiding center of the particle. In this system the particle has a spiral path, gaining energy from the field by a betatron like mechanism (Fig. 3).

The gain in energy ε of a particle in this system is:

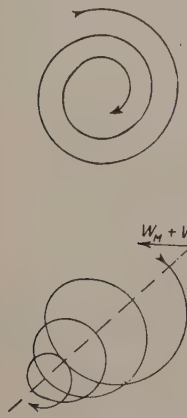

Fig. 3. – Path of a plasma particle in a varying magnetic field: *a*) in the guiding center co-ordinate system; *b*) in the laboratory system.

$$(23) \qquad d\varepsilon = d\left(\frac{\frac{1}{2}mu^2}{B}B\right) = d(\mu_m B)\,,$$

where μ_m is the magnetic moment defined by

$$(24) \qquad \mu_m = \frac{\frac{1}{2}mu^2}{B} = \frac{1}{2}qRu\,.$$

It is known that μ_m is a constant of motion. It is worth-while to point to a physical meaning of this fact. Eq. (24) can be written in the form:

$$(25) \qquad \frac{1}{2}\frac{mu^2}{B} = \frac{1}{2\pi}\frac{q^2}{m}(R^2\pi B)\,.$$

The bracket expression is just the flux enclosed by the particle orbit. This means that the magnetic field is frozen into each particle orbit, providing again an equation of the form of eq. (7). The « material curve » is in this case the particle path.

It follows from eq. (23) making use of the constancy of μ_m:

$$(26) \qquad d\varepsilon = \mu_m dB\,,$$

which gives denoting initial values corresponding to $t = 0$ by index 0:

$$(27) \qquad \frac{\varepsilon}{\varepsilon_0} = \frac{B}{B_0}\,,$$

that is the particle energy is proportional to the magnetic field measured always at the guiding center. From this follows furthermore that picking up a bunch of particles in a small region of the plasma, the energy of each of them will be multiplied by the same factor B/B_0 owing to the same drift component in the direction of grad $|\boldsymbol{B}|$. (Only $\boldsymbol{w}_E$ possesses such a component in the first order theory). Consequently a local Maxwellian distribution is maintained during the process, for the particles concerned, and eq. (27) can be

written in terms of two dimensional kinetic temperatures,

$$\frac{T}{T_0} = \frac{B}{B_0}. \tag{28}$$

The plasma pressure can be expressed in terms of the particle density and of the kinetic temperatures (see for example eqs. (34) and (25) of paper I):

$$p = p_0\left(\frac{n}{n_0}\right)\left(\frac{T}{T_0}\right). \tag{29}$$

For each material region for which eq. (22) is valid we may rewrite it in the form:

$$\frac{V}{V_0} = \frac{n_0}{n} = \frac{B_0}{B}. \tag{30}$$

By combination of eqs. (28), (29) and (30), we obtain directly the equations of state:

$$V = V_0\left(\frac{p_0}{p}\right)^{\frac{1}{\varkappa}}. \tag{31}$$

and

$$T = T_0\left(\frac{p}{p^0}\right)^{\frac{1}{\varkappa}}. \tag{32}$$

Considering that the number of degrees of freedom is $f = 2$, eq. (10) yields $\varkappa = 2$ and it can be readily seen, that eqs. (31) and (32) agree with eqs. (9) and (11) respectively. With this, the formal analogy between the behaviour of plasmas in the two entirely different limiting cases in completed.

Otherwise it can be shown in a more general—although more formal—way that a tenuous plasma in arbitrary electromagnetic fields obeys always adiabatic equations of state (6). Namely for such a plasma the Boltzmann equation reduces to $\mathrm{d}f/\mathrm{d}t = 0$. This means that the density of any bunch of particles in the six dimensional $(\boldsymbol{r}, \boldsymbol{v})$ space remains constant in the course of motion. As the probability, of finding a particle in the six dimensional unit volume is uniform in the phase space the thermodynamic probability of any bunch of particles remains constant as well. This is equivalent to the conservation of entropy of any part of the system, proving our statement that the thermodynamic transformations concerned are adiabatic.

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(6) This was derived recently on different lines by I. B. BERNSTEIN: *Phys. Rev.*, **109**, 10 (1958).

We turn now anew to the problem of the behaviour of an initially field free plasma in a growing magnetic field, treated in Sect. 2 in the magneto-hydrodynamic approximation. In the first stages of the process, when the applied field is yet small, the particle paths become slightly bent. Only in the later stages are the previous considerations applicable, in case that the following restrictions are satisfied:

1) The cyclotron radius of any plasma particle is much smaller than the linear dimensions of the vessel. In this case, the great majority of particles are confined by the field and not by the material container.

2) The usual condition for the first order treatment is satisfied. This requires that the variation of the magnetic field should be small inside a cyclotron orbit.

If the magnetic field is sufficiently large (and R small) the treatment above becomes applicable, provided that the time rates are sufficiently small for the quasi-stationary treatment. We may inquire now about the variation of various plasma characteristics in terms of the external magnetic field $B(\infty, t)$.

The combination of eqs. (28), (29) and (30) leads to:

$$p = p_0 \frac{B^2}{B_0^2}, \tag{33}$$

which can be rewritten in the form:

$$p(\xi, t) = p(\xi_0, 0) \frac{B^2(\xi, t)}{B^2(\xi_0, 0)}, \tag{34}$$

where the new notations express clearly the fact that quantities belonging to different instants refer also to different space co-ordinates. The co-ordinate ξ equals ξ_0 at $t = 0$, and follows the motion of the guiding centres of the particles.

To express the variables concerned in terms of the external magnetic field $B(\infty, t)$, we use the integrated form of the pressure balance equation (12), which yields for any point ξ at the instant t:

$$\frac{B^2(\xi, t)}{2\mu} + p(\xi, t) = \frac{B^2(\infty, t)}{2\mu}. \tag{35}$$

For $t = 0$ eq. (35) reduces to:

$$\frac{B^2(\xi_0, 0)}{2\mu} + p(\xi_0, 0) = \frac{B^2(\infty, 0)}{2\mu}. \tag{36}$$

By combining eqs. (34), (35) and (36) we readily obtain:

$$\frac{B(\xi, t)}{B(\xi_0, 0)} = \frac{B(\infty, t)}{B(\infty, 0)} . \tag{37}$$

This is a very important result. It means that, increasing the external field to its ν fold value during the time t, *each particle* experiences a ν fold increase of the magnetic field in its own moving co-ordinate system. If the initial distributions at $t = 0$ are known, each quantity may be readily obtained as functions of $B(\infty, t)$ only. From eqs. (34) and (37) we have:

$$p(\xi, t) = p(\xi_0, 0) \frac{B^2(\infty, t)}{B^2(\infty, 0)} . \tag{38}$$

Combining this with eq. (32) we obtain:

$$T(\xi, t) = T(\xi_0, 0) \frac{B(\infty, t)}{B(\infty, 0)} \tag{39}$$

or with eq. (31) the volume occupied by any bunch of particles is:

$$V(\xi, t) = V(\xi_0, 0) \frac{B(\infty, 0)}{B(\infty, t)} . \tag{40}$$

Appendix A

Supposing we produced somehow a partially or totally ionized « cool » plasma in a cylindrical container, then both temperature and pressure are enhanced by application of a growing magnetic filed.

Assuming an initial temperature of $T_0 \approx 10^5$ °K and pressure of $p_0 \approx 10^{-3}$ atm and a magnetic field of about $B_{max} = 16\,000$ gauss peak value (actually a rather conservative estimate), the corresponding material pressure after the compression is roughly:

$$p_{max} = \frac{B_{max}^2}{2\mu_0} \approx 10^6 \frac{N}{m^2} \approx 10 \text{ atm} . \tag{41}$$

Considering a « two dimensional » compression ($\varkappa = 2$), the corresponding temperature is from eq. (32):

$$T_{max} = T_0 \left(\frac{p_{max}}{p_0} \right)^{\frac{1}{2}} \approx 10^7 \text{ °K} . \tag{42}$$

The plasma volume is during the compression compressed to

$$V_{\min} = V_0 \left(\frac{p_0}{p_{\max}}\right)^{\frac{1}{\varkappa}} \approx 10^{-2} V_0 . \tag{43}$$

A « three dimensional » compression yields with $\varkappa = \frac{5}{3}$

$$T_{\max} \approx 4 \cdot 10^6 \,^0\mathrm{K} \qquad \text{and} \qquad V_{\min} \approx 4 \cdot 10^{-3} V_0 .$$

Which $\varkappa$ is to be chosen, depends upon the compression time τ.

The « self collision time », the average time required to reduce any lack of anisotropy in the velocity distribution is [5]:

$$t_0 = \frac{11.4\, A^{\frac{1}{2}} T^{\frac{3}{2}}}{n Z^4 \ln \Lambda} , \tag{44}$$

where $A = m/m_{\text{proton}}$, $n =$ particle/cm^3, and $\ln \Lambda$ is a slowly varying function of T and n, and it may be taken $\ln \Lambda \approx 10$. In a plasma of deuterium ions $Z = 1$ and $A = 2$. Assuming a linear increase of B with time, and $\varkappa = 2$:

$$B = \frac{t}{\tau} B_{\max} ; \quad T = \frac{t}{\tau} T_{\max} ; \quad n = \frac{t}{\tau} n_{\max} .$$

The probability of a considerable change of the distribution function by collisions during the compression, is:

$$\int_0^\tau \frac{\mathrm{d}t}{t_c} = \frac{\ln \Lambda n_{\max}}{11.4 \cdot \sqrt{2}\, T_{\max}^{\frac{3}{2}}} \tau^{\frac{1}{2}} \int_0^\tau \frac{\mathrm{d}t}{t^{\frac{1}{2}}} = \frac{\ln \Lambda \sqrt{2}\, n_{\max}}{11.4\, T_{\max}^{\frac{3}{2}}} \tau . \tag{45}$$

The assumption of a « two dimensional compression » is justified until this quantity is much less than 1, or

$$\tau \ll \frac{11.4\, T_{\max}^{\frac{3}{2}}}{\ln \Lambda \sqrt{2}\, n_{\max}} \approx 4 \cdot 10^{-6} \text{ s} . \tag{46}$$

For electrons however, this quantity is much less because $A \approx 1/2\,000$. In this case we obtain the condition:

$$\tau \ll 10^{-7} \text{ s} .$$

Therefore it is obvious that in case of $\tau \ll 10^{-7}$ s the compression is two dimensional, when $\tau \gg 4 \cdot 10^{-6}$ s, it is three dimensional. In the intermediate range, the situation is rather complicated. It is generally impossible for the two types of particles to undergo a different type of compression, because this would result in net space charges and immense electrical field strengths. The two types of particles are furthermore coupled by the common magnetic field determined by currents of both types of particles. For this « mixed » compression $\frac{5}{3} < \varkappa < 2$.

For $\tau \gg 4 \cdot 10^{-6}$ there is a strong collision interaction, and the magneto-hydrodynamical model describes the phenomena. There exists an upper limit for τ, because of the diffusion of magnetic field into the plasma, due to the finite conductivity. We can make an estimation for this with the help of eq. (6). If we approximate $\nabla^2 B \approx B/L^2$ (where L is of the order of magnitude of linear dimensions) and substitute L^2 for S, we obtain:

$$\frac{1}{\mu\sigma}\frac{B}{L^2}L^2 \approx \frac{\mathrm{d}\Phi}{\mathrm{d}t}. \tag{47}$$

Since $BL^2 \approx \Phi$ we get

$$\frac{\mathrm{d}\Phi}{\Phi} \approx \frac{\mathrm{d}t}{\mu\sigma L^2}. \tag{48}$$

The time constant for diffusion is therefore $\vartheta \approx \mu\sigma L^2$. The conductivity of a plasma is $\sigma = T^{\frac{3}{2}}/653$ [5]. In case of a cylindrical geometry L^2 is proportional to V. With

$$T = T_{\max}\left(\frac{p}{p_{\max}}\right)^{\frac{2}{5}} \qquad \text{and} \qquad L^2 = L^2_{\min}\left(\frac{p_{\max}}{p}\right)^{\frac{2}{5}}.$$

We obtain for ϑ:

$$\vartheta = \mu\frac{T^{\frac{3}{2}}_{\max}L^2_{\min}}{653}. \tag{49}$$

This shows that ϑ is the same constant in every phase of the compression. An initial value of $L_0 = 0.1$ m results *e.g.* $L^2_{\min} \approx 4 \cdot 10^{-5}\ \mathrm{m}^2$ and $\vartheta \approx 6 \cdot 10^{-4}$ s.

In the collisionless case the size is limited by the condition $L \gg R$. With $\varkappa = 2$, and the same cylindrical geometry: $L^2 = L^2_{\min}(B_{\max}/B)$. It can be easily shown that L/R is a constant during the compression, since combining $R = mv/qB$ with eq. (24) we obtain by means of an elementary calculation:

$$\frac{L}{R} = L_{\min}q\sqrt{\frac{B_{\max}}{2\mu_m m}}. \tag{50}$$

For deuterons with $B_{\max} = 16\,000$ gauss, $R_{\min} \approx 4$ mm. A final plasma diameter of some centimeters yields an initial size of some 10 cm, which is quite reasonable.

The assumption of an adiabatic compression is justified only when energy, losses are negligible. Indeed, assuming a compression time of about 1 μs, the loss by recombination radiation per particle is of the order of 10^{-2} eV, the bremsstrahlung radiation is 10^{-3} eV whereas the energy gain during the compression is $(10^2 \div 10^3)$ eV. The neglect of particle loss by recombination is also well justified, the rate during the contraction amounting to some 10^{-7} recomb/particle.

Appendix B

Finally we investigate the case of high compression velocities, when the velocity of the « magnetic wall » reaches the order of magnitude of the particle velocities in the plasma. We excluded this case in our former treatment in order to avoid « non-quasi-stationary » processes.

We investigate in the following the magneto-hydrodynamic model in the ideal case of no magnetic field penetration. The phenomena in this case do not differ from the simple process of compressing a gas in a cylinder with help of a moving piston, only the piston is substituted by the « magnetic wall ».

Assuming now a piston in periodic operation of compressing and expanding a gas, it is easy to see that this process can be regarded as quasi-stationary only as long as the piston velocity V is much lower than the molecular velocities of the gas v. When however V reaches the order of magnitude of v, the energy gain of the gas in the compression period—when the piston is running towards the gas—is larger, while the energy loss by expansion—when the piston is running away—is smaller than in the quasi-stationary case. The net effect arising from this irreversible phenomenon is, that in a cyclic operation the gas is heated to ever increasing temperatures ([7]).

A statistical explanation of this fact is the following: Given an ensemble of microscopic particles interacting with each other and with the piston. The average kinetic energy of the piston, $\langle \frac{1}{2} M V^2 \rangle$ corresponds to a very high temperature, due to the high mass M. The particles tend to reach the temperature equilibrium with the practically infinitely « hot » piston, by obtaining kinetic energy from it ([8]).

We investigate on a simple example how this process works. Assuming a one dimensional problem, with not interacting particles, and two synchronously moving walls with a momentary distance $L(t)$, the wall velocity is

$$V = \frac{1}{2} \frac{\mathrm{d}L}{dt} . \tag{51}$$

If the particle velocity v is small compared to V, we obtain for the time of flight between two reflections:

$$\tau = \frac{L + V\tau}{v} \tag{52}$$

or

$$\tau = \frac{L}{v - V} . \tag{53}$$

([7]) The « magnetic pumping » mentioned in ref. ([4]) seems to be similar to this method.

([8]) G. Kalman, L. Pocs, G. Schmidt and K. Simonyi: *Periodica Politechnica Budapest* (Electr. Eng. Series), **1**, 53 (1957).

The number of reflections in the time dt equals dt/τ while the gain in velocity is

$$dv = -2V\frac{v-V}{L}\,dt. \tag{54}$$

In case of $r/V \to \infty$ denoting r in this asymptotic case by u, we have

$$\frac{du}{u} = -2\frac{V}{L}\,dt = -\frac{dL/dt}{L}\,dt. \tag{55}$$

After integration we obtain

$$u = u_0\frac{L_0}{L}, \tag{56}$$

where u_0 and L_0 correspond to the same $t = t_0$. As $u^2 = u_0^2(T/T_0)$ and $Lp/T = L_0p_0/T_0$ from eq. (56) follows

$$T = T_0\left(\frac{p}{p_0}\right)^{\frac{2}{3}}. \tag{57}$$

As for a one dimensional case $\varkappa = 3$, eq. (57) equals eq. (11). For the quasi-stationary case we put $v = u \cdot y$ in eq. (54) and obtain

$$y\,du + u\,dy = -2V\frac{uy}{L}\,dt + \frac{2V^2}{L}\,dt; \tag{58}$$

with help of eqs. (55) and (56) it follows:

$$\int_1^y dy = \frac{2}{L_0u_0}\int_0^t V^2\,dt. \tag{59}$$

From this we obtain finally

$$v = \frac{u_0L_0}{L} + \frac{2}{L}\int_0^t V^2\,dt. \tag{60}$$

We see that in the irreversible case v is always greater than in eq. (56). In our case of « Magnetic piston » we control the pressure instead of L. Expressing again L and v by p and T we obtain from eq. (60)

$$A^2\left(\frac{p}{T}\right)^2 + 2AC\left(\frac{p}{T}\right) + C = T \tag{61}$$

applying the following notations:

$$A = \frac{T_0^{\frac{3}{2}}}{p_0} \quad \text{and} \quad C = \frac{2}{L}\int_0^t V^2 \, \mathrm{d}t \, .$$

As A and C are both positive, it is obvious, that the same T corresponds now to a smaller p/T and therefore to a smaller p then in the reversible case when $C = 0$.

RIASSUNTO (*)

Si esamina il comportamento di un plasma posto in un campo magnetico esterno considerando due casi limite: 1) il caso idromagnetico ideale ($\sigma \to \infty$); 2) il caso privo di collisioni. Restringendo i calcoli a geometrie cilindriche generali ($\partial/\partial z = 0$) e a processi quasi stazionari si ottiene per i due casi un sistema completo di equazioni del plasma. Vi sono incluse tra altre equazioni di stato anche equazioni per plasma tenuo. Esiste una notevole somiglianza tra i due sistemi di equazioni riferentisi a condizioni fisiche totalmente differenti. Le equazioni si lasciano prontamente risolvere per esprimere ogni quantità macroscopica di plasma in termini dell'intensità del campo magnetico esterno. Si descrive in breve l'estensione delle nostre considerazioni a processi quasi non stazionari. Ciò conduce a un fenomeno che suggerisce nuovi metodi per aumentare le temperature del plasma.

(*) *Traduzione a cura della Redazione.*

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On the Inversion Problem for a Klein-Gordon Wave Equation.

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Summary. — We treat in this paper the problem of obtaining the static potential which corresponds to given resonance parameters in the case of a Klein-Gordon wave equation. Our results are an extension of those already found in the non relativistic case. We restrict the class of our potentials to those vanishing outside a finite range a. The knowledge of the resonance parameters allows to obtain an approximate solution of the so called Gel'fand-Levitan equation which provides the desired potential.

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1. – We consider a Klein-Gordon equation for S-waves:

$$\varphi''(E, x) + [K^2 - 2E\,V(x) + V^2(x)]\varphi(E, x) = 0\,, \tag{1}$$

with the purpose of deriving the potential $V(x)$ when its spectrum is known; the most direct way is provided by the so-called Gel'fand and Levitan equation corresponding to eq. (1).

This reads (*):

$$K_{\alpha\beta}(x, y) = f_{\alpha\beta}(x, y) - \int_0^\infty K_{\alpha\gamma}(x, t)\, f_{\gamma\beta}(t, y)\, \mathrm{d}t\,, \tag{2}$$

(*) In this equation α, β have two values 1, 2. Its first derivation is due to Corinaldesi [1], who has limited himself to $\alpha=1$ and $\beta=1, 2$. We have followed here the formulation of Verde [2], by which $K_{\alpha\beta}$ is a 2×2 matrix. The reason for the appearance of it is due, as we will precise below, to the fact that the Klein-Gordon operator is not self-adjoint and the completeness relations require two-component eigenwaves.

(1) E. Corinaldesi: *Nuovo Cimento*, **11**, 468 (1954).

(2) M. Verde: *The inversion problem in wave mechanics and dispersion relations*, to be of published on Nuclear Physics.

$f_{\alpha\beta}(x, y)$ contains the whole information, being defined by

$$f_{\alpha\beta}(x, y) = \frac{1}{\pi}\int_c k\,\mathrm{d}E\left[1 - \frac{1}{G(E)}\right]\Phi_{0\alpha}(E, x)[\sigma_1\Phi_0(E, y)]_\beta . \tag{3}$$

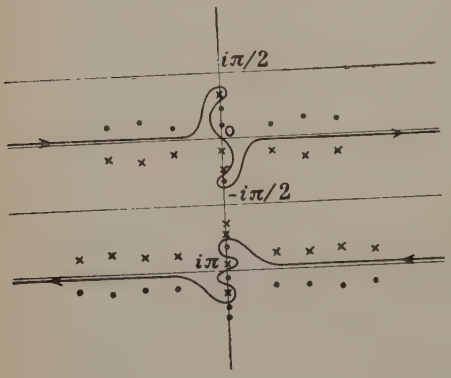

Fig. 1. – Path of integration. The complex plane represented here is the ξ-plane. E and k are single valued functions of ξ: $E = m\cosh(\xi/m)$; $k = m\sinh(\xi/m)$. The whole E plane is mapped in the region $-3\pi/2 \leqslant \mathrm{Im}\ \xi \leqslant \pi/2$. • Zeros of $f(K)$; x Zeros of $f(-K)$.

Here $\Phi_0(E, x)$ is the wave matrix (defined later) whose first component is the eigensolution of (1) in absence of the potential normalized so that for $x \to 0$, $\Phi_0(E, x) \sim x$. $G(E)$ is related to the so-called spectral function as we shall specify later on. $\int_c$ means that the path of integration has to be taken as in Fig. 1.

From eq. (2) the potential $V(x)$ follows, being

$$\frac{K_{11}(x, x)}{K_{12}(x, x)} = \frac{1}{2}[V(x) + V(0)] . \tag{4}$$

We wish to approximate the $f_{\alpha\beta}$, once a set of resonance parameters is experimentally obtained, so that it yields a simple solution for eq. (3).

2. – In agreement with JOST and KOHN [3] we define the two independent solutions of (1) $f(k, x)$ and $f(-k, x)$ with the asymptotic behaviour

$$f(k, x)_{x\to\infty} \sim \exp[-ikx] , \qquad f(-k, x)_{x\to\infty} \sim \exp[ikx] .$$

The solution of (1) which for $x \to 0$ goes like x can be expressed as:

$$\varphi(E,x) = \frac{1}{2ik}[f(k)\,f(-k, x) - f(-k)\,f(k, x),]$$

$f(k) = f(k, 0)$. The S matrix and the phase shift are defined by

$$S(k) = \exp[2i\delta] = \frac{f(k)}{f(-k)} .$$

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[3] R. JOST and W. KOHN: *Dan. Mat. Fys. Medd.*, **27**, no. 9 (1953).

Two solutions of (1) corresponding to energies E and E' satisfy the quasi-orthogonality relation

$$(5) \qquad \int_0^\infty [E + E' - 2V(x)]\,\varphi(E,x)\,\varphi(E',x)\,\mathrm{d}x = \delta(E - E')\,G(E)\,\frac{\pi}{k}\,,$$

where $G(E) = f(k)\,f(-k)$ and $|E| > m$.

We know that $\varphi(E, x)$ is an entire function of E of type x (*). $f(k)$ however is a two valued function of k, having two branching points at $k = \pm im$. The same happens to E considered as function of k. This suggests that $G(E)$ might be an one valued function of E because it is even in k and it has the same branching points. A close examination of $G(E)$ shows that this is indeed true.

The set of functions $\varphi(E, x)$ is two times complete, *i.e.* it can be used to expand two functions. This is clear when one considers the classical limit, in which one retains only the positive energy solutions of (1) which form a complete set in the interval $0 \cdots \infty$. It is customary to define two component wave functions $\Phi(E, x)$ defined by

$$(6) \qquad \Phi(E, x) = \begin{pmatrix} \varphi(E, x) \\ (E - V)\,\varphi(E, x) \end{pmatrix}.$$

On this set the scalar product is defined as

$$(7) \qquad [\Phi(E, x), \Phi(E', x)] = \int_0^\infty \Phi_\alpha(E, x)[\sigma_1\,\Phi(E', x)]_\alpha\,\mathrm{d}x\,,$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The set (6) is orthogonal in the sense of (7). It is also complete because of the expansion

$$(8) \qquad \delta_{\alpha\beta}(x - x') = \int_{-\infty}^{+\infty} \mathrm{d}\varrho(E)\,\Phi_\alpha(E, x)[\sigma_1\,\Phi(E, x')]_\beta\,.$$

(*) Let $F(k)$ be an entire function of k and $M(h)$ the maximum modulus of $F(k)$ in the circle of center $k=0$ and radius h. The order of $F(k)$ is defined by

$$\limsup_{h\to\infty} \frac{\log\log M(h)}{\log h} = \varrho\,.$$

The limit $\limsup_{h\to\infty} h^{-\varrho} \lg M(h)$ is the type of $F(k)$. See R. P. Boas: *Entire functions* (New York, 1954).

Here the spectral function is defined as

$$\frac{\mathrm{d}\varrho(E)}{\mathrm{d}E} = \frac{k}{\pi G(E)} \qquad \text{if} \quad |E| > m;$$

$$\frac{\mathrm{d}\varrho(E)}{\mathrm{d}E} = \sum_l \left[2\int_0^\infty (E_l - V)|\varphi_l|^2 \,\mathrm{d}x\right]^{-1} \delta(E - E_l) \qquad \text{for} \quad |E| < m,$$

E_l and φ_l denote the energy and eigenfunction of the l-th bound state.

A more compact form of (8) can be obtained by going into the plane of the complex energy

$$\delta_{\alpha\beta}(x - x') = \frac{1}{\pi}\int_c \frac{k\mathrm{d}E}{G(E)} \Phi_\alpha(E, x)[\sigma_1 \Phi(E, x')]_\beta .$$

3. – Many of the results of the non-relativistic analysis can be generalized without any difficult to the Klein-Gordon equation. It is obvious that $G(E)$ is an entire function of E of type $2a$, where a is the range of the potential. The same approach as in [4] yields quite similar results concerning the distribution of the zeros of $G(E)$ and the corresponding expansion of this function into an infinite product. In dealing with these properties we observe that it is by far convenient to use as independent variable the energy E instead of the wave number k, just because $\varphi(E, x)$ is not a uniform function of k, while it is uniform in E.

We are now interested in the classification of the zeros of $G(E)$. The real zeros of $G(E)$ belonging to the interval $-m \cdots m$ imply the existence of particle or antiparticle bound states. The remaining zeros either belong to some zero norm pathological states (*) or they are the energies of the so-called resonance states (in infinite number) or finally a few of them define the virtual states. Apart from the bound states and the zero norm states the solutions which correspond to the other roots are not acceptable from a strict quantum mechanical point of view because of their non summability; yet in many practical cases (*e.g.* in nuclear physics) the complex energy resonance states are treated as true states with some individuality. We want to investigate the completeness of the set S of $\Phi(E, x)$ which correspond to the zeros of $G(E)$.

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(4) T. REGGE: *Nuovo Cimento*, **8**, 671 (1958); *Nuovo Cimento*, **9**, 491 (1958).

(*) If the width and depth of the potential are large enough some states of vanishing norm may occur (5). Although their spatial behaviour is quite correct their energy is complex. We shall not consider this case as it leads to unwanted difficulties.

(5) L. I. SCHIFF, H. SNYDER and J. WEINBERG: *Phys. Rev.*, **57**, 315 (1940).

We shall find that this set S is complete in the interval $0 \cdots 2a$. This is the starting point for an expansion of the functions $f_{\alpha\beta}$ of (3) in terms of the set S, in order to obtain a form for them which is practical for computation.

Let $\Phi(E_n, x)$ be any function of the set S. If

$$\sum_{1}^{2}{}_{\alpha} \int_{0}^{H} \Phi_\alpha(E_n, x)[\sigma_1\, \xi(x)]\, \mathrm{d}x = 0$$

for every n implies $\xi(x) = 0$ in $0 \cdots H$, then S is closed in $0 \cdots H$.

We introduce the entire function $I(E)$ of type H defined by

$$I(E) = \int_{0}^{H} \Phi_\alpha(E, x)[\sigma_1\, \xi(x)]_\alpha\, \mathrm{d}x\,,$$

$I(E)$ can be factorized as follows: $I(E) = G(E)R(E)$, where $R(E)$ is an entire function. Since the type of $G(E)$ is $2a$ the type of $I(E)$ is at least $2a$. Therefore unless $H > 2a$ we must take $I(E) = 0$, $R(E) = 0$. The conclusion is that the set S is complete in the interval $0 \cdots 2a$. The independence of the set follows from the existence of another auxiliary set of functions $\tau_n(x)$ defined by

$$\frac{G(E)}{E - E_n} = \int_{0}^{\infty} \Phi_\alpha(E, x)[\sigma_1\, \tau_n(x)]_\alpha\, \mathrm{d}x\,. \tag{9}$$

Eq. (9) is a generalized Paley-Wiener theorem and its detailed proof requires some care and some additional considerations. From (9) we deduce

$$\tau_n(x) = \frac{1}{\pi} \int_{c} k\, \mathrm{d}E\, \frac{G(E)}{E - E_n}\, \Phi(E, x)\,.$$

It is evident from (9) that $\tau_m(x)$ is orthogonal in the sense of (7) to all $\Phi_n(E, x)$ but $\Phi_m(E, x)$, and it vanishes for $x > 2a$. Any relation of the kind

$$\sum_{0}^{\infty}{}_{n} \lambda_n \Phi(E_n, x) = 0$$

holding in the interval $0 \cdots 2a$ implies therefore that all $\lambda_n = 0$.

The completeness and independence of the set S in $0 \cdots 2a$ imply the same for the set $\Phi_0(E_n, x)$ of the unperturbed solutions. The set

$$\vartheta_n(x) = \frac{1}{\pi} \int k\, \mathrm{d}E\, \frac{G(E)}{E - E_n}\, \Phi_0(E, x)$$

satisfies orthogonality properties:

$$\int_0^{2a} \vartheta_{n\alpha}(x)\,[\sigma_1\,\Phi_0(E_m, x)]_\alpha\,\mathrm{d}x = \delta_{nm} G'(E_m)\,.$$

4. – The completeness of the set $\Phi_0(E_n, x)$ allows to obtain an approximate formula for the kernel $f_{\alpha\beta}$ of the Gel'fand and Levitan integral equation. We can expand $f_{\alpha\beta}$ using the set of $\Phi_0(E_n, x)$:

$$f_{\alpha\beta}(x, y) = \sum_n \omega_{n\alpha}(x)\,[\sigma_1\,\Phi_0(E_n, y)]_\beta\,,$$

where $\omega_{n\alpha}(x)$ is given by:

$$\omega_{n\alpha}(x) = \frac{1}{G'(E_n)} \int_0^{2a} \vartheta_{n\beta}(y) f_{\alpha\beta}(xy)\,\mathrm{d}y =$$

$$= \frac{1}{\pi G'(E_n)} \int_0^{2a} \vartheta_{n\beta}(y)\,\mathrm{d}y \int_c k\,\mathrm{d}E \left[1 - \frac{1}{G(E)}\right] \Phi_{0\alpha}(E, x)\,[\sigma_1\,\Phi_0(E, y)]_\beta =$$

$$= \frac{1}{\pi G'(E_n)} \int_c k\,\mathrm{d}E \left[1 - \frac{1}{G(E)}\right] \Phi_{0\alpha}(E, x)\,\frac{G(E)}{E - E_n}\,.$$

Taking into account that $\vartheta_n(x)$ satisfies the equation

$$\vartheta_n(x) + f_0(-k_n, x) = \frac{1}{\pi} \int_c k\,\mathrm{d}E\,\frac{G(E) - 1}{E - E_n}\,\Phi_0(E_n, x)\,,$$

where $f_0(-k_n, x)$ is defined by $f_0(k_n, x) = \begin{pmatrix} \exp\,[ik_n x] \\ (E_n - V)\exp\,[ik_n x] \end{pmatrix}$ we obtain for $f_{\alpha\beta}(x, y)$ the expansion

$$f_{\alpha\beta}(x, y) = \sum_n \frac{1}{G'(E_n)}\,[\vartheta_n(x) + f_0(-k_n, x)]_\alpha\,[\sigma_1\,\Phi_0(E_n, y)]_\beta\,.$$

But

$$\sum_n \frac{1}{G'(E_n)}\,\vartheta_{n\alpha}(x)\,[\sigma_1\,\Phi_0(E_n, y)]_\beta = \delta_{\alpha\beta}(x - y) = 0 \qquad \text{if } x > y$$

so that the expansion assumes the simple form

$$f_{\alpha\beta}(x, y) = \sum_n \frac{1}{G'(E_n)}\,f_{0\alpha}(-k_n, x)\,[\sigma_1,\, \Phi_0(E_n, y)]_\beta\,.$$

If $x < y$ we can exchange x and y in the formula in virtue of the symmetry.

With this expansion we can construct $f_{\alpha\beta}(x, y)$ from the experimental data. This form of $f_{\alpha\beta}$, as it is evident, is especially suited for the integration of eq. (2), since our approximation of $f_{\alpha\beta}$ consists in a finite sum of products of functions of the variable x or y only.

* * *

I wish to thank Prof. M. VERDE for many helpful discussions and Prof. T. REGGE who suggested this problem.

RIASSUNTO (*)

In questo lavoro trattiamo il problema di ottenere il potenziale statico corrispondente a dati parametri di risonanza nel caso di un'equazione d'onda di Klein-Gordon. I nostri risultati sono un ampliamento di quelli già trovati per il caso non relativistico. Restringiamo la classe dei nostri potenziali a quelli che si annullano oltre un range finito a. La conoscenza dei parametri di risonanza ci consente di ottenere una soluzione approssimata della cosidetta equazione di Gel'fand-Levitan che fornisce il potenziale richiesto.

(*) *Traduzione a cura della Redazione.*

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The Polarization of Knock-on Electrons Arising from Scattering by Polarized μ-Mesons.

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(ricevuto l'11 Agosto 1958)

Summary. — An experiment is proposed for the direct measurement of the helicity of μ-mesons arising from π decay by studying the polarization of the high energy electrons produced in the collision of the polarized μ-mesons with unpolarized electrons. Polarization effects become significant only for high energy incident μ's (of the order of a few GeV). The separation of a polarized beam of μ-mesons from the decays in flight of high energy π's is in principle made easier by the fact that the laboratory momenta of μ-mesons which are emitted in the forward and backward directions in the π rest system differ by about 42% as a consequence of the relativistic transformation from the center of mass to the laboratory system. Expressions in the laboratory frame are also given of the cross section for scattering polarized electrons by polarized μ-mesons. In the C.M. frame, the results agree with those calculated by Bincer.

1. – Introduction.

It is well known that knowledge of the helicity of μ mesons produced in the decay of the π would provide considerable information concerning the coupling involved in the decay interactions. Techniques used to measure the helicity of electrons in β-decay, which utilize spin effects in Rutherford scattering and bremsstrahlung, can probably not be successfully applied to μ mesons because of their relatively large mass (1). Suggestions haver ecently been

(*) Supported in part by the U. S. Air Force through the OSR.

(1) J. Franklin and B. Margolis (*Phys. Rev.*, **109**, 525 (1958)) discuss the possibility of utilizing Mott scattering of μ-mesons by nuclei to determine the μ helicity.

made to study the μ helicity by observing the asymmetry of the electron angular distribution in the β-decay following the capture of polarized μ mesons in the ^{12}C present in nuclear emulsion ([2]). Depolarization effects of various kinds, and the low energy of the recoiling ^{12}B nucleus (used in identifying the event), make this experiment a difficult one to perform. Another possibility which has been discussed ([3]) involves the knock-on scattering of polarized μ mesons on aligned electrons. It is suggested that μ mesons, appropriately selected to be only from forward decays in the rest system of the π, be made incident on a block of iron magnetized so that a fraction of the electron spins are successively parallel and antiparallel to the μ beam. The cross-section for knock-on electrons is then to be compared for the two cases.

We are also concerned with the knock-on process in the present note. If one scatters unpolarized electrons by polarized μ mesons, the final electrons are polarized. We obtain expressions for this polarization in Sect. **2**. Since spin dependent effects arise from a spin spin interaction which is of much shorter range than the Coulomb interaction between electron and μ meson, the spin dependent terms can be expected to become important only for μ mesons which have laboratory energies large compared to the rest mass of the meson. The high energy electrons arising from a knock-on scattering by μ mesons of a few GeV are sufficiently polarized that the polarization of the electrons in the resulting shower ([4]) may possibly be detected. The low cross-section for μ-e scattering at the high incident meson energies required may make detection of the polarization difficult.

A partially polarized beam of high energy μ mesons can be obtained by selecting a fraction of those which have appeared from the decay of a beam of energetic π mesons of uniform momentum. For example, π mesons with a total energy of 3 GeV result in 3.003 GeV/c momentum for the μ mesons which were emitted in the forward direction in the π center of mass system, and 1.726 GeV/c for those emitted backward.

In the Appendix, we list the results in the laboratory frame for μ-e knock-on cross-sections when both particles are initially polarized. These results, when transformed to the center of mass system, agree with those of BINCER ([3]).

2. – Calculations and results.

We consider the final state density matrix for the process in which a μ meson, of four-momentum $p_1 = (\boldsymbol{p}_1, i\varepsilon_1)$, kinetic energy E and mass M ($\varepsilon_1 = E + M$), scatters an electron of mass m at rest ($p_2 = (\boldsymbol{0}, im)$). The

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([2]) J. D. JACKSON, S. B. TREIMAN and H. W. WYLD Jr.: *Phys. Rev.*, **107**, 327 (1957)
([3]) A. M. BINCER: *Phys. Rev.*, **107**, 1434 (1957).
([4]) K. W. MCVOY and F. J. DYSON: *Phys. Rev.*, **106**, 1360 (1957).

final state is a μ meson of four-momentum p_1' and an electron of four-momentum p_2', and polarization $\boldsymbol{l}_2'$: An energy E' ($E' = \varepsilon_2' - m$) is imparted to the electron. We define the four-vectors [5]

$$s_1 = P_1 \frac{\varepsilon_1}{M} (\hat{\boldsymbol{p}}, i\beta) , \tag{1}$$

and

$$s_2' = \left(\boldsymbol{l}_2' + \frac{(\boldsymbol{l}_2' \cdot \boldsymbol{p}_2')}{m(\varepsilon_2' + m)} \boldsymbol{p}_2', i \frac{\boldsymbol{l}_2' \cdot \boldsymbol{p}_2'}{m} \right) . \tag{2}$$

P_1 is the percentage polarization of the μ and is positive or negative, depending on whether it is parallel or antiparallel to its direction of incidence: $\hat{\boldsymbol{p}}$ is a unit vector in the direction of incidence, and β is v/c for the incoming μ. A simple calculation, which utilizes the positive energy projection operator

$$\Lambda^+(s, p) = (4m)^{-1}(1 + i\gamma_5 \gamma \cdot s)(m - i\gamma \cdot p) \tag{3}$$

for taking into account the specific spin orientation, leads to a final state density matrix

$$\varrho_f = r_0^2 m^2 \frac{\delta(\varepsilon_1' + \varepsilon_2' - \varepsilon_1 - \varepsilon_2) \, \mathrm{d}^3 \boldsymbol{p}_2'}{[(p_1 \cdot p_2)^2 - m^2 M^2]^{\frac{1}{2}} \varepsilon_1' \varepsilon_2' (p_1' - p_1)^4} \, Y . \tag{4}$$

In the expression above, r_0 is the classical electron radius, and Y is given by:

$$Y = Y_0 + Y_s^f , \tag{5}$$

where

$$Y_0 = (m^2 + M^2)(m^2 + p_2 \cdot p_2') + (p_1 \cdot p_2)^2 + (p_1' \cdot p_2)^2 ; \tag{6}$$

$$Y_s^f = - mM[s_1 \cdot s_2'(p_1 - p_1')^2 + s_1 \cdot (p_1 - p_1') \, s_2' \cdot (p_2 - p_2')] . \tag{7}$$

The differential cross-section and polarization at final electron energy E' are given by [6]

$$\sigma_0 = \mathrm{Tr} \, \varrho_f , \tag{8}$$

and

$$\boldsymbol{P} = \frac{\mathrm{Tr} \, \varrho \, \boldsymbol{\sigma}}{\mathrm{Tr} \, \varrho} , \tag{9}$$

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[5] H. A. TOLHOEK: *Rev. Mod. Phys.*, **28**, 277 (1956).

[6] L. WOLFENSTEIN and J. ASHKIN: *Phys. Rev.*, **85**, 947 (1952).

where it is understood that we let $\boldsymbol{l}_2' \to \boldsymbol{\sigma}$ and that the traces involve summations or integrations over all final variables with the exception of the electron energy.

If we specialize to laboratory co-ordinates, we find that the differential cross-section has the well-known form (7)

$$\sigma_0(E, E')\,\mathrm{d}E' = \frac{2\pi m r_0^2}{\beta E'^2}\left[1 - \beta^2\frac{E'}{E_m'} + \frac{1}{2}\left(\frac{E'}{E+M}\right)^2\right]\mathrm{d}E', \tag{10}$$

where E_m' is the maximum energy which can be imparted to the electron. This value of E' is reached for electrons scattered in the forward direction. Equation (10) is the cross-section calculated for unpolarized particles.

The component of polarization along the incident direction, expressed in laboratory co-ordinates, is given by

$$\mathcal{P}(E,E') = \frac{P_1\left(\dfrac{E'}{E+M}\right)\left[1 + E'\left(\dfrac{2m-E'}{2m+E'}\dfrac{1}{E_m'} + \dfrac{1}{2(E+M)}\right) - \dfrac{4m}{E'+2m}\right]}{1 - \beta^2\dfrac{E'}{E_m'} + \dfrac{1}{2}\left(\dfrac{E'}{E+M}\right)^2}. \tag{11}$$

The average polarization is of course entirely along the incident direction. If only knock-on electrons between energies of E_m' and $\eta E_m'$ are detected, this polarization is defined by:

$$\Gamma_\eta = \frac{\mathcal{P}_\eta P_1}{\Sigma_\eta^0} = \frac{P_{11}^- \displaystyle\int_{\eta E_m'}^{E_m'} \sigma_0(E, E')\,\mathcal{P}(E, E')\,\mathrm{d}E'}{\displaystyle\int_{\eta E_m'}^{E_m'} \sigma_0(E, E')\,\mathrm{d}E'}. \tag{12}$$

The behavior of Γ_η and Σ_η^0 as functions of E is presented in the Table.

* * *

We would like to thank L. Madansky for suggesting this experiment as well as the means of obtaining polarized beams of high energy μ mesons.

(7) See, for example, B. Rossi: *High Energy Particles* (New York, 1952) p. 16.

APPENDIX

We summarize here the results obtained for the differential cross-sections when polarized μ-mesons scatter electrons polarized along or opposite to the direction of incident μ-meson momentum. In the expression for the density matrix, Eq. (4), Y is replaced by Y', where

$$Y' = Y_0 + Y_s' , \tag{A.1}$$

and

$$Y_s' = -mM[s_1 \cdot s_2 (p_1 - p_1')^2 + s_1 \cdot (p_1 - p_1')\, s_2 \cdot (p_2 - p_2')] . \tag{A.2}$$

In Eq. (A.2) above, s_2 is the four vector

$$s_2 = P_2(\hat{\boldsymbol{p}}, 0) , \tag{A.3}$$

and P_2 is the percentage polarization of the initial electron, positive if parallel, negative if antiparallel to the direction of the incident μ. The differential cross-section in the laboratory co-ordinates has the form

$$\sigma(E, E')\,\mathrm{d}E' = [\sigma_0(E, E') + \sigma_s(E, E')]\,\mathrm{d}E' , \tag{A.4}$$

where

$$\sigma_s(E, E') = -\frac{2\pi m r_0^2}{\beta E'^2} P_1 P_2 \frac{E'}{E + M}\left(1 - \frac{E'}{E_m'} + \frac{1}{2}\frac{E'}{E + M}\right) . \tag{A.5}$$

In order to get a measure of the spin dependent part of the cross-section, we define Δ_η, where

$$\Delta_\eta = \frac{-(P_1 P_2)^{-1} \int_{\eta E_m'}^{E_m'} \sigma_s(E, E')\,\mathrm{d}E'}{\int_{\eta E_m'}^{E_m'} \sigma_0(E, E')\,\mathrm{d}E'} . \tag{A.6}$$

The behavior of Δ_η as a function of E is presented in the table.

TABLE I. – *Table of results.*

E/M	E'_m/m	β^2	$\Sigma^0_{.5}$ (mb)	$\Gamma_{.5}$ (%)	$\Delta_{.5}$ (%)	$\Sigma^0_{.9}$ (10^{-2} mb)	$\Gamma_{.9}$ (%)	$\Delta_{.9}$ (%)
10	217	.992	.73	6.4	6.6	1.87	13.7	14.4
20	729	.9977	.22	12.4	12.5	.46	31.5	34.2
30	1480	.9996	.11	18.1	18.3	.25	46.9	48.6

RIASSUNTO (*)

Si propone un esperimento per la misura diretta dell'elicità dei mesoni μ derivanti dal decadimento π per mezzo dello studio della polarizzazione degli elettroni di alta energia prodotti nella collisione dei mesoni μ polarizzati con elettroni non polarizzati. Gli effetti della polarizzazione appaiono significativi solo per μ incidenti di alta energia (dell'ordine di qualche GeV). La separazione di un fascio di mesoni μ polarizzati dai decadimenti in volo di π di alta energia è facilitato in linea di principio dal fatto che gli impulsi nel sistema del laboratorio dei mesoni μ emessi in avanti e all'indietro nel sistema di riposo dei π differiscono di circa il 42% come conseguenza della trasformazione relativistica dal sistema del centro di massa a quello del laboratorio. Si danno anche espressioni valevoli nel sistema del laboratorio per la sezione d'urto degli elettroni polarizzati dallo scattering su mesoni μ polarizzati. Nel sistema del centro di massa i risultati si accordano con quelli calcolati da Bincer.

(*) *Traduzione a cura della Redazione.*

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The Radiative Pion-Nucleon Scattering in Static Theory.

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(ricevuto il 18 Agosto 1958)

Summary. — The radiative pion-nucleon scattering is studied in static theory treating one meson exactly, and the other one in the one-meson approximation. Explicit expressions for the rescattering corrections are obtained.

1. – Introduction.

In this paper we shall apply the static meson theory of Chew in order to evaluate the matrix elements for radiative pion nucleon scattering, *i.e.* for the process:

$$\pi + \mathrm{N} \to \pi + \mathrm{N} + \gamma \,.$$

This problem has already been treated by Cutkosky (1) who has been able to treat the incoming and outgoing pions in a symmetrical way. His method, however, leads to a very difficult evaluation of the rescattering corrections.

In our paper we are able to obtain explicit expressions for these corrections. The method treats one pion exactly and the other one in a one-meson approximation. It is especially adequate for situations in which only one of the two pions is in the (33) resonant state. The case in which both pions are on

(1) R. E. Cutkosky: *Phys. Rev.*, **109**, 209 (1958).

the (33) resonance will be treated separately in a forthcoming publication. In this case indeed the photon energy k is quite small and therefore, following Low (2), one can derive a rigorous low energy theorem.

2. – Radiative pion-nucleon scattering matrix element.

The matrix element of our process is

$$\mathcal{H}_k(q, p) = \left(\Psi_q^{(-)}, \int (-\boldsymbol{j}\boldsymbol{A}_k^*)\, \mathrm{d}v \Psi_p^{(+)}\right), \tag{1}$$

where p and q represent the initial and the final meson respectively, k is the emitted photon and the other symbols are the same as in CHEW-LOW's paper (3).

As in the C.L. paper, we put:

$$\boldsymbol{j} = \boldsymbol{j}_V + \boldsymbol{j}_S + \boldsymbol{j}_\pi\,, \tag{2}$$

with the condition:

$$\begin{cases} [a, \boldsymbol{j}_V + \boldsymbol{j}_S] = [a_q^*, \boldsymbol{j}_V + \boldsymbol{j}_S] \quad = 0\,, \\ (\Psi_0, [\boldsymbol{j}_V + \boldsymbol{j}_S]\Psi_0) = (\Psi_0, \boldsymbol{j}\Psi_0) = 0\,. \end{cases} \tag{3}$$

Then by the same argument of C.L. we get for the vector current matrix element:

$$\mathcal{H}_k^V(q, p) = -\frac{\mu_p - \mu_N}{2} \frac{F_1(k^2)}{f_r} \left(\Psi_q^{(-)}, \frac{f_r^0 i \tau_3 \boldsymbol{\sigma} \wedge \boldsymbol{k} \cdot \boldsymbol{\epsilon}}{(2k)^{\frac{1}{2}}} \Psi_p^{(+)}\right) = \tag{4}$$

$$= -\frac{\mu_p - \mu_N}{2} \left(\frac{\omega_i}{k}\right)^{\frac{1}{2}} \frac{F(k^2)}{v(k)} (\Psi_q^{(-)}, V_u^0 \Psi_p^{(+)})\,,$$

where V_u^0 is given by

$$V_u^0 = i f_r \tau_3 \frac{\boldsymbol{\sigma} \cdot \boldsymbol{u}}{\sqrt{2\omega_u}} v(u)$$

and corresponds to a meson of momentum $\boldsymbol{u} = \boldsymbol{k} \wedge \boldsymbol{\epsilon}$ and isotopic variable 3.

We have thus to study the matrix element:

$$(\Psi_q^{(-)}, V_u^0 \Psi_p^{(+)})\,.$$

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(2) F. E. LOW: *Phys. Rev.*, **110**, 974 (1958).

(3) G. F. CHEW and F. E. LOW: *Phys. Rev.*, **101**, 1579 (1955). In the following extensive use will be made of the results and of the notations of this work, that will be quoted as C.L.

It can be expressed in the following way:

$$(5)\qquad (\Psi_q^{(-)},\ V_u^0\Psi_p^{(+)}) = (\Psi_0,\ a_q V_u^0\Psi_p^{(+)}) - \left(\Psi_0 V_q^{0+}\frac{1}{H-\omega_q-i\varepsilon}V_u^0\Psi_p^{(+)}\right) =$$

$$= \delta_{pq}(\Psi_0,\ V_u^0\Psi_0) - \sum_n \frac{(\Psi_0,\ V_u^0\Psi_n^{(-)})(\Psi_n^{(-)}V_q^{0+}\Psi_p^{(+)})}{E_n+\omega_q-\omega_p-i\varepsilon} - \sum_n \frac{(\Psi_0,\ V_q^{0+}\Psi_n^{(-)})(\Psi_n^{(-)},\ V_u^0\Psi_p^{(+)})}{E_n-\omega_q-i\varepsilon},$$

where the identity:

$$(6)\qquad a_q\Psi_p^{(+)} = \delta_{pq}\Psi_0 - (H+\omega_q-\omega_p-i\varepsilon)^{-1}V_q^0\Psi_p^{(+)}$$

has been used.

We now introduce the new function

$$(\Psi_q^{(-)},\ V_u^{(0)}\Psi_p^{(+)}) - \delta_{pq}(\Psi_0,\ V_u^{(0)}\Psi_0) = (q^-,\ p^+)_u .$$

From (5) we obtain for $(q^-,\ p^+)_u$ the following equation in the one-meson approximation:

$$(7)\qquad (q^-,\ p^+)_u = -\frac{(\Psi_0,\ V_u^0\Psi_0)(\Psi_0,\ V_q^{0+}\Psi_p^{(+)})}{\omega_q-\omega_p-i\varepsilon} - \frac{(\Psi_0,\ V_u^0\Psi_p^{(-)})(\Psi_0,\ V_q^{0+}\Psi_0)}{\omega_q} +$$

$$+\frac{(\Psi_0,\ V_q^{0+}\Psi_0)(\Psi_0,\ V_u^0\Psi_p^{(+)})}{\omega_p} - \frac{(\Psi_0,\ V_q^{0+}\Psi_p^{(-)})(\Psi_0,\ V_u\Psi_0)}{\omega_p-\omega_q-i\varepsilon} - \sum_r \frac{(\Psi_0,\ V_u^0\Psi_r^{(-)})(r^-,\ p^+)_q}{\omega_r+\omega_q-\omega_p-i\varepsilon} -$$

$$-\sum\frac{(\Psi_0,\ V_q^{0+}\Psi_r^{(-)})(r^-,\ p^+)_u}{\omega_r-\omega_q-i\varepsilon} - \sum_r \frac{(\Psi_0,\ V_u^{0+}\Psi_{rp}^{(-)})(\Psi_r^{(-)},\ V_q^{0+}\Psi_0)}{\omega_r+\omega_q} -$$

$$-\sum\frac{(\Psi_0,\ V_q^{0*}\Psi_{rp}^{(-)})(\Psi_r^{(-)},\ V_u^0\Psi_0)}{\omega_r+\omega_p-\omega_q-i\varepsilon}.$$

We point out that in order to obtain in (7) all the terms with the zero and one meson in the intermediate states, we need to consider in (5) also the two-mesons terms. In fact the two-mesons matrix element $(\Psi_q V_u \Psi_{rs})$ contains a δ-function term, as one easily recognize by writing for this matrix element an equation similar to the (5). When this δ-function is substituted in (5) it leads to a one-meson term ([4]).

We should now study the matrix element $\mathcal{H}_k^s(q,\ p)$ due to the scalar current density. However, since we have the same reason as in photoproduction of neglecting this term in the final result, it will not have any consideration in our paper.

([4]) L. RODBERG: *Phys. Rev.*, **106**, 1090 (1957).

We turn out to the matrix element

$$(8)\quad \mathcal{H}_k^{\pi}(q,p)=\left(\Psi_q^{(-)},\int(-\boldsymbol{j}_\pi\cdot\boldsymbol{A}_k^*)\,\mathrm{d}v\,\Psi_p^{(+)}\right)=\left(\Psi_0\left[a_q,\int\mathrm{d}v\,(-\boldsymbol{j}_\pi\cdot A_k^*)\right]\Psi_p^{(+)}\right)+$$

$$+\left(\Psi_0,\int\mathrm{d}v(-\boldsymbol{j}_\pi\cdot A_k^*)\,a_q\Psi_p^{(+)}\right)-\sum_n(\Psi_0V_q^{0+}\Psi_n^{(-)})\frac{\mathcal{H}_k^{\pi}(n,p)}{E_n-\omega_q-i\varepsilon}\,.$$

Using the relation (6) to express $a_q\Psi_p^{(+)}$ the second term can be written

$$\left(\Psi_0,\int\mathrm{d}v\,(-\boldsymbol{j}_\pi\cdot A_k^*)a_q\Psi_p^{(+)}\right)=-\sum_n\left(\Psi_0,\int\mathrm{d}v\,(-\boldsymbol{j}_\pi\cdot A_k^*)\Psi_n^{(-)}\right)\frac{(\Psi_n^{(-)}V_q^{0+}\Psi_p^{(+)})}{E_n+\omega_q-\omega_p-i\varepsilon}\,,$$

since from (3) follows:

$$(9)\qquad (\Psi_0,\boldsymbol{j}_\pi\Psi_0)=0\,.$$

Finally Eq. (8) takes the form:

$$(10)\quad \mathcal{H}_k^{\pi}(q,p)=\left(\Psi_0,\left[a_q,\int\mathrm{d}v\,(-\boldsymbol{j}_\pi\cdot A_k^*\right]\Psi_p^{(+)}\right)-$$

$$-\sum_n\left(\Psi_0,\int\mathrm{d}v\,(-\boldsymbol{j}_\pi\cdot A_k^*)\,\Psi_n^{(-)}\right)\frac{(\Psi_n^{(-)}V_q^{0+}\Psi_p^{(+)})}{E_n+\omega_q-\omega_p-i\varepsilon}-\sum(\Psi_0,V_q^{0+}\Psi_n^{(-)})\frac{\mathcal{H}_k^{\pi}(n,p)}{\omega_n-\omega_q-i\varepsilon}\,,$$

where there are no contributions to the sum for $n=0$ because of (9).

We shall define

$$(11)\qquad \mathcal{H}_k^{0}(q,p)=\left(\Psi_0,\left[a_q,\int\mathrm{d}v\,(-\boldsymbol{j}_\pi\cdot A_k^*)\right]\Psi_p^{(+)}\right).$$

The commutator appearing in this expression is the same as the one entering in the C.L. work on photoproduction problem; therefore we can write

$$(12)\qquad \mathcal{H}_k^{0}(q,p)=\mathcal{H}_{\mathrm{int}}^{0}(q,p)+\mathcal{H}_M^{0}(q,p)\,,$$

where

$$\mathcal{H}_{\mathrm{int}}^{0}(q,p)=\frac{e_r f_r^0}{(4k\omega_q)^{\frac{1}{2}}}\,(\Psi_0,[\tau_2\delta_{q1}-\tau_1\delta_{q2}]\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\,\Psi_p^{(+)}),$$

$$\mathcal{H}_M^{0}(q,p)=\frac{2ie_r\boldsymbol{q}\cdot\boldsymbol{\epsilon}}{(8k\omega_q\omega_{q+k})^{\frac{1}{2}}}\left\{(\delta_{q1}\delta_{p2}-\delta_{q2}\delta_{p1})\delta_{p,\,q+k}(\Psi_0,\Psi_0)\right\}+$$

$$+\frac{2\omega_{q+k}}{\omega_p^2-\omega_{q+k}^2+i\varepsilon}\,(\Psi_0[V_{2q+k}^{0+}\delta_{q1}-V_{1q+k}^{0+}\delta_{q2}]\psi_p^{(+)})$$

$\mathcal{H}^0_{\text{int}}$ is coming from the interaction current and $\mathcal{H}^1_M$ from the conventional meson current defined by Eq. (3) of C.L.'s work.

If we limit ourselves to the one-meson approximation, Eq. (10) becomes an integral equation for the matrix element $\mathcal{H}^\pi_k(q, p)$, since $\mathcal{H}^0_k(q, p)$ is known and the remaining term is given by the product of the solution of Eq. (5) times a matrix element of photoproduction.

We observe here that $\mathcal{H}^0_I(q, p)$ does not contribute any more to the rescattering correction in the integral equation, since it is a S-wave in the k variable. It can be expressed in a suitable way by means of the scattering matrix.

We have

$$\mathcal{H}^0_{\text{int}}(q, p) = \frac{4\pi i}{(8k\omega_q\omega_p)^{\frac{1}{2}}} \sum_{IJ\prime} \varepsilon_{3\beta\gamma} P_I(\gamma, \alpha)\, P_J(\boldsymbol{\epsilon}, \boldsymbol{p})\, h_{IJ}(\omega_p)\,, \tag{13}$$

if α and β are the isotopic index of the p and q mesons respectively and where

$$\left|\begin{array}{l} P_{\frac{1}{2}}(\boldsymbol{\epsilon}, \boldsymbol{p}) = \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\,\boldsymbol{\sigma}\cdot\boldsymbol{p}\,, \\ P_{\frac{3}{2}}(\boldsymbol{\epsilon}, \boldsymbol{p}) = 3\,\boldsymbol{\epsilon}\cdot\boldsymbol{p} - \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\,\boldsymbol{\sigma}\cdot\boldsymbol{p}\,, \\ P_{\frac{1}{2}}(\gamma, \alpha) = \frac{1}{3}\,\tau_\gamma\,\tau_\alpha\,, \\ P_{\frac{3}{2}}(\gamma, \alpha) = \delta_{\gamma\alpha} - \frac{1}{3}\,\tau_\gamma\,\tau_\alpha\,. \end{array}\right. \tag{14}$$

Moreover, we can neglect the term

$$\sum_r \left(\Psi_0, \int \mathrm{d}v\,(-\boldsymbol{j}_\pi\cdot\boldsymbol{A}^*_k)\,\Psi^{(-)}_r\right) \frac{(\Psi^{(-)}_r V^{0+}_q \Psi^{(+)}_p)}{\omega_r + \omega_q - \omega_p - i\varepsilon}\,,$$

because of the smallness of such a matrix element in comparison with other inhomogeneous terms, and since unless the photon energy is bigger than the meson rest mass, the deniminator cannot go to zero.

To recapitulate, we have:

$$\mathcal{H}^\pi_k(\boldsymbol{q}, \boldsymbol{p}) = \mathcal{H}^0_I(q, p) + \mathcal{H}^M_k(q, p)\,, \tag{15}$$

where $\mathcal{H}^M_k(q, p)$ is defined by the integral equation

$$\mathcal{H}^M_k(q, p) = \mathcal{H}^0_M(q, p) - \sum_r (\Psi_0 V^{0+}_q \Psi^{(-)}_r)\, \frac{\mathcal{H}^N_k(r, p)}{\omega_r - \omega_q - i\varepsilon}\,. \tag{16}$$

Therefore the total radiative pion nucleon scattering matrix-element is given by:

$$\mathcal{H}_k(q, p) = \mathcal{H}^0_I(q, p) + \mathcal{H}^V_k(q, p) + \mathcal{H}^M_k(q, p)\,.$$

3. – Solutions of the integral equations.

We study in this section the solutions of Eq. (7) and (16). Starting wtih (7) we observe that the last three terms can be neglected since two of them have a denominator which never vanishes, the denominator of the remaining one can only go to zero if the photon energy is larger than the meson rest-mass, and this case will not be considered as will be discussed in Sect. **4**. Using the projectors ${}_tP_T$ and ${}_jP_J$ defined in Appendix A we can write:

$$(17) \qquad (q^-, p^+)_u = \sum_{tTJ} R^{TJ}_{tj}\, {}_tP_T\, {}_jP_J \frac{f_r 4\pi i v(q)\, v(p)\, v(u)}{(8\omega_q\omega_r\omega_u)^{\frac{1}{2}}} ,$$

where R^{TJ}_{tj} satisfy the integral equation:

$$(18) \qquad R^{TJ}_{tj}(\omega_q, \omega_p) = \frac{A^{TJ}_{tj}}{\omega_q} + \frac{B^{TJ}_{tj}}{\omega_p - \omega_q} + \frac{1}{\pi}\int \frac{h^*_{tj}(\omega_r)\, R^{TJ}_{tj}(\omega_r, \omega_p)}{\omega_r - \omega_q - i\varepsilon}\, v^2(r) r^3\, \mathrm{d}\omega_r .$$

The matrices A^{TJ}_{tj} and B^{TJ}_{tj} are also defined in Appendix A. For the solution of such an equation, we use the RODBERG [5] result which in our case reads:

$$(19) \qquad R^{TJ}_{tj}(\omega_q, \omega_p) = \left[\frac{A^{TJ}_{tj}(\omega_p)}{\lambda_{tj}} + \frac{B^{TJ}_{tj}(\omega_p)}{h_{tj}(\omega_p)} \frac{\omega_q}{\omega_p} \frac{1}{\omega_q - \omega_p}\right] h_{tj}(\omega_q)$$

where for

$$t_j = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 3 & 3 \end{bmatrix}; \qquad \lambda_{tj} = \frac{2}{3} f^2 \begin{bmatrix} -4 \\ -1 \\ +2 \end{bmatrix} .$$

We now turn our attention to Eq. (16). Since the equation is linear, for the sake of convenience we write:

$$(20) \qquad \mathcal{H}^M_k(q, p) = \mathcal{H}^A_k(q, p) + \mathcal{H}^k_B(q, p) ,$$

where

$$(20') \qquad \mathcal{H}^A_k = \mathcal{H}^{0A}_k(q, p) - \sum_r \frac{(\Psi_0 V^{0+}_q \Psi^{(-)}_r)}{(\omega_r - \omega_q - i\varepsilon)} \mathcal{H}^A_k(r, p) ,$$

$$(20'') \qquad \mathcal{H}^B_k = \mathcal{H}^{0B}_k(q, p) - \sum_r \frac{(\Psi_0 V^{0+}_q \Psi^{(-)}_r)}{(\omega_r - \omega_q - i\varepsilon)} \mathcal{H}^B_k(r, p) ,$$

$$(21') \qquad \mathcal{H}^{0A}_k(q,p) \frac{2ie\boldsymbol{\epsilon}\cdot\boldsymbol{q}}{(8k\omega_q\omega_{q+k})^{\frac{1}{2}}} \delta_{p,q+k}\varepsilon_{3\beta\alpha}(\Psi_0, \Psi_0) ,$$

$$(21'') \qquad \mathcal{H}^{0B}_k(q, p) = \frac{2ie\,\boldsymbol{\epsilon}\cdot\boldsymbol{q}}{(4k\omega_q)^{\frac{1}{2}}} \frac{\sqrt{2\,\omega_{q+k}}}{\omega_p^2 - \omega^2_{q+k} + i\varepsilon} \varepsilon_{3\beta\gamma}(\Psi_0 V^{0+}_{\gamma,q+k} \Psi^{(+)}_p) .$$

---

[5] See equation (28) of ref. [4].

Here α and β stay for the isotopic index of p and q ,mesons. Starting with (20′) in order to deal with integral equation without δ-function it is convenient to introduce a new function $C_k(q, p)$ defined as:

$$C_k(q, p) = \mathcal{H}_k^A(q, p) - \mathcal{H}_k^{0A}(q, p) . \tag{22}$$

The resulting equation for $C_k(q, p)$ is:

$$C_k(q, p, r) = -\frac{2ie\,\boldsymbol{\epsilon}\cdot\boldsymbol{p}}{(8k\omega_r\omega_p)^{\frac{1}{2}}}\frac{(\Psi_0, \Psi_0)}{\omega_r - \omega_q - i\varepsilon}\sum_\gamma \varepsilon_{3\gamma\alpha}(\Psi_0, V_q^+\Psi_{\gamma,r}^{(-)}) - \\ - \sum_n \frac{(\Psi_0, V_q^0\Psi_n^{(-)})}{\omega_n - \omega_q - i\varepsilon} C_k(n, p, r) . \tag{23}$$

where $\boldsymbol{r} = \boldsymbol{p} - \boldsymbol{k}$.

It can be easily seen that (23) is satisfied by:

$$C_k(q, p, r) \equiv C_k^{\beta\alpha}(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{r}) = \\ = \frac{4\pi\, r(q)\, r(p)}{(8\omega_q\omega_p k)^{\frac{1}{2}}}\frac{1}{\omega_r}\boldsymbol{p}\cdot\boldsymbol{\epsilon}\, e(\Psi_0, \Psi_0)\sum_{IJ}\langle\beta|P_I t^3|\alpha\rangle P_J(\boldsymbol{q}, \boldsymbol{r})\, C_{IJ}(\omega_q, \omega_r) \tag{24}$$

if $C_{IJ}(\omega_q, \omega_r)$ is solution of the equation

$$C_{IJ}(\omega_q, \omega_r) = \frac{1}{\omega_r - \omega_q - i\varepsilon} h_{I,J}^*(\omega_r) + \frac{1}{\pi}\int \frac{v^2(n) n^3\, \mathrm{d}\omega_n}{\omega_n - \omega_q - i\varepsilon} h_{I,J}^*(\omega_n)\, C_{IJ}(\omega_n, \omega_r) . \tag{25}$$

This equation is of same type as (18); the explicit expression for C_{IJ} is:

$$C_{IJ}(\omega_q, \omega_r) = \frac{h_{IJ}^*(\omega_r)}{h_{IJ}(\omega_r)}\frac{\omega_q}{\omega_r}\frac{1}{\omega_r - \omega_q} h_{IJ}(\omega_q) . \tag{26}$$

Finally considering (20″), we have to observe that the kernel will rescatter only the p-waves. Then we can consider in the $\mathcal{H}_k^{0B}$ only that part which contains p-waves in q variable.

We separate our amplitude into magnetic dipole and electric quadrupole parts. Furthermore, we can label our magnetic and electric amplitudes with the quantum number of initial and final pion-nucleon system. The electric quadrupole transition of initial $j = \frac{1}{2}$ state can only lead to final $j = \frac{3}{2}$ states. All the remaining transitions can lead from one of the four initial states to one of the four final states of the pion-nucleon system.

We can therefore set

$$\mathcal{H}_k^B(q, p) = \text{non } p\text{-wave part or } \mathcal{H}_k^{0B}(q, p) + \\ + \frac{8\pi e\, v(q)\, v(p)}{(8k\omega_q\omega_p)^{\frac{1}{2}}}\sum_{iIj}\Big\{Q_{j,i}^{\frac{3}{2},I}(q, p)[\boldsymbol{q}\cdot\boldsymbol{k}\, P_{\frac{1}{2}}(\boldsymbol{\epsilon}, \boldsymbol{p}) + \boldsymbol{q}\cdot\boldsymbol{\epsilon}\, P_{\frac{1}{2}}(\boldsymbol{k}, \boldsymbol{p})] + \\ + Q_{j,i}^{\frac{3}{2},I}(q, p)\tfrac{1}{3}[P_j(\boldsymbol{q}, \boldsymbol{k})\, P_{\frac{3}{2}}(\boldsymbol{\epsilon}, \boldsymbol{p}) + P_j(\boldsymbol{q}, \boldsymbol{\epsilon})\, P_{\frac{3}{2}}(k, \boldsymbol{p})] + M_{j,i}^{\frac{1}{2},I}(q, p)\, i[P_j(\boldsymbol{q}, \boldsymbol{k}\wedge\boldsymbol{\epsilon})\boldsymbol{\sigma}\cdot\boldsymbol{p}] + \\ + M_{j,i}^{\frac{3}{2},I}(q, p)\tfrac{1}{3}[P(\boldsymbol{k}, \boldsymbol{q})\, P_{\frac{3}{2}}(\boldsymbol{\epsilon}, \boldsymbol{p}) - P_j(\boldsymbol{q}, \boldsymbol{\epsilon})\, P_{\frac{3}{2}}(\boldsymbol{k}, \boldsymbol{p})]\Big\}\langle\beta|P_i t^3 P_I|\alpha\rangle . \tag{27}$$

The indices have the following meaning: the upper ones are the total angular momentum (at left) and the total isotopic spin (at right) of initial state; the lower indices are in the same order the angular momentum and isospin of pion-nucleon system in final state. The Q and the M are the electric quadrupole and the magnetic dipole amplitudes.

The equations for these amplitudes become:

$$(28') \quad Q^{J;I}_{j;i}(q, p) = G_Q(q, p, k) h_{IJ}(\omega_p) + \frac{1}{\pi} \int \frac{n^3 v^2(n)\, \mathrm{d}\omega_n}{\omega_n - \omega_q - i\varepsilon} h^*_{ij}(\omega_n) Q^{J;I}_{j;i}(n, p) ,$$

$$(28'') \quad M^{J;I}_{j;i}(q, p) = \Lambda^J_j G_M(q, p; k) h_{IJ}(\omega_p) + \frac{1}{\pi} \int \frac{n^3 v^2(n)\, \mathrm{d}\omega_n}{\omega_n - \omega_q - i\varepsilon} h^*_{ij}(\omega_n) M^{J;I}_{j;i}(n, p) ,$$

$$\Lambda \equiv \begin{matrix} J = \frac{1}{2} & J = \frac{3}{2} & \\ \left\{ \begin{matrix} \frac{2}{3} & 1 \\ -\frac{1}{3} & 1 \end{matrix} \right. & \begin{matrix} j = \frac{1}{2} \\ j = \frac{3}{2} \end{matrix} \end{matrix}$$

The explicit expressions for G_Q, G_M are

$$(29) \quad \begin{cases} G_M = \begin{cases} \dfrac{3}{16kq}\left[(x^2-1)\log\dfrac{1+x}{1-x} - 2x - \pi i(x^2-1)\right] & \text{for } |x|<1, \\ \dfrac{3}{16kq}\left[(x^2-1)\log\dfrac{x+1}{x-1} - 2x\right] & \text{for } |x|>1, \end{cases} \\ G_Q = \begin{cases} -\dfrac{3}{8k^2}\left[(x^3-x)\log\dfrac{1+x}{1-x} - 2x^2 + \dfrac{4}{3} - i\pi(x^3-x)\right] - G_M & \text{for } |x|<1, \\ -\dfrac{3}{8k^2}\left[(x^3-x)\log\dfrac{1+x}{x-1} - 2x^2 + \dfrac{4}{3}\right] - G_M & \text{for } |x|>1, \end{cases} \end{cases}$$

and

$$x = \frac{p^2 - q^2 - k^2}{2kq} .$$

The solution of equations (28) was given by OMNES [6]:

$$(30') \quad Q^{J;I}_{j;i}(q, p) = \left[G_Q(q, p, k) h_{IJ}(\omega_p) \cos\delta_{ij}(\omega_q) + \right.$$
$$\left. + \frac{1}{\pi} \exp[\varrho_{ij}(\omega_q)] P \int \frac{G_Q(\xi, p, k) \sin\delta_{ij}(\xi)}{\xi - \omega_q} \exp[-\varrho_{ij}(\xi)]\, \mathrm{d}\xi \right] \exp[i\delta_{ij}(\omega_q)] ,$$

$$(30'') \quad M^{J;I}_{j;i}(q, p) = \text{similar expression with } G_Q \text{ replaced by } \Lambda^J_j G_M$$

and

$$\varrho_{ij}(\omega_q) = \frac{1}{\pi} P \int \frac{\delta_{ij}(\xi)}{\xi - \omega_q} \mathrm{d}\xi .$$

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(6) R. OMNÈS: *Nuovo Cimento*, **8**, 316 (1958).

4. – Discussion and conclusions.

The results of our investigations show that the scattering amplitudes for $\pi + p \to p + \pi + \gamma$ are most important, and therefore could have some chance to be experimentally observed, when either one or both mesons are on the 33 resonance. Our method, which treats one of the two mesons exactly, is particularly appropriate for the first situation: the pion which is to be treated exactly is the one in the resonant state.

In principle our method applies equally well when either the initial or the final pion is the resonant one.

However we think the more interesting case is when the initial meson is a resonant state: this makes the total energy smaller and therefore the static approximation and the one meson approximation more reliable.

Concerning the shape of the photon spectrum our method is appropriate for the study of its intermediate energy part. As already pointed out in the introduction, the low energy part can be accounted exactly.

On the other hand, it is not easy to evaluate the error for $k \simeq \mu$. The high energy part needs further investigations.

* * *

The greater part of this work has been done while one of the authors (B.B.) was fellow at Theoretical Group of CERN in Geneva.

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APPENDIX A

In the final state we have to couple three angular momenta ($\frac{1}{2}$ of nucleon spin, 1 of meson q of the vector $\boldsymbol{u} = \boldsymbol{k} \wedge \boldsymbol{\epsilon}$) in order to get $\frac{1}{2}$ or $\frac{3}{2}$ of the initial pion nucleon system (the initial meson is p).

It is easy to verify that if we choose a coupling scheme in which the q meson is coupled to the nucleon to give the partial angular momentum j the four operators of total angular momentum J and partial j which we shall indicate as ${}_jP_J(u, q, p)$ are:

$$\text{(A.1)} \qquad \begin{cases} {}_{\frac{1}{2}}P_{\frac{1}{2}} = \Lambda_j P_j(\boldsymbol{q}, \boldsymbol{k} \wedge \boldsymbol{\epsilon})\, \boldsymbol{\sigma} \cdot \boldsymbol{p}\,, \\ {}_{\frac{3}{2}}P_{\frac{3}{2}} = i[P_j(\boldsymbol{q}, \boldsymbol{\epsilon}) P_{\frac{3}{2}}(\boldsymbol{k}, \boldsymbol{p}) - P_j(\boldsymbol{q}, \boldsymbol{k}) P_{\frac{3}{2}}(\boldsymbol{\epsilon}, p)]\,; \end{cases}$$

where $\Lambda_j = \begin{matrix} \\ \left[\begin{matrix} 2 \\ -1 \end{matrix}\right] \end{matrix} \begin{matrix} j \\ \frac{1}{2} \\ \frac{3}{2} \end{matrix}$,

P_j are the usual operators as defined by CHEW.

The corresponding ones for the isotopic spin, which we shall indicate as ${}_tP_T$, are:

$$ {}_tP_T = \langle \beta | P_t t^3 P_T | \alpha \rangle , \tag{A.2} $$

where $t^i_{kl} = -i\varepsilon_{ikl}$.

In (A.1) we introduce the matrix B_j in order to preserve the symmetry between angular momentum and isotopic spin. The matrix A^{TJ}_{tj} and B^{TJ}_{tj} can then be written in the form (all the terms are omitted but the 33 resonant one):

$$ A^{11}_{11} = \frac{12}{27} h^*_{33} ; \quad A^{11}_{13} = -\frac{8}{27} h^*_{33} ; \quad A^{11}_{33} = \frac{4}{27} h^*_{33} , $$

$$ A^{13}_{11} = -\frac{20}{27} h^*_{33} ; \quad A^{13}_{13} = -\frac{8}{27} h^*_{33} ; \quad A^{13}_{33} = \frac{4}{27} h^*_{33} , $$

$$ A^{13}_{31} = \frac{10}{27} h^*_{33} ; \quad A^{33}_{11} = \frac{25}{27} h^*_{33} - 3 h_{33} ; \quad A^{33}_{13} = \frac{10}{27} h^*_{33} $$

$$ A^{33}_{33} = \frac{4}{27} h^*_{33} , $$

$$ B^{11}_{11} = B^{11}_{13} = B^{13}_{11} = B^{13}_{13} = B^{13}_{31} = 0 , $$

$$ B^{11}_{33} = \frac{4}{3} h^*_{33} ; \quad B^{13}_{33} = -\frac{2}{3} h^*_{33} ; \quad B^{33}_{11} = -\frac{4}{3} h_{33} , $$

$$ B^{33}_{13} = \frac{2}{3} h_{33} ; \quad B^{33}_{33} = \frac{h^*_{33} - h_{33}}{3} . $$

The remaining elements are given by the relations:

$$ A^{RS}_{rs} = A^{SR}_{sr} ; \qquad B^{RS}_{rs} = B^{SR}_{sr} . $$

The h_{ij} are function of ω_p and are defined as:

$$ h_{ji}(\omega_p) = \exp[i\delta_{ij}] \sin \delta_{ij} / p^3 v^2(p) . $$

RIASSUNTO (*)

Si studia nel quadro della teoria statica lo scattering radiativo pione-nucleone trattando un mesone esattamente, e l'altro nell'approssimazione di un solo mesone. Si ottengono espressioni esplicite per le correzioni di riscattering.

(*) *Traduzione a cura della Redazione.*

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An Alternative Method of Quantization: the Existence of Classical Fields.

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Summary. — A new method of quantization, applicable to an arbitrary physical system, is presented. For systems with a classical analog, the introduction of quantized canonical variables proceeds without the elimination of the corresponding unquantized variables (although the Liouville equation is in general modified). These remain as legitimate physical observables having the same expectation value as and no greater dispersion than their quantized counterparts. Their (non-negative definite) joint distribution in phase space then determines the state of the system. In the case of bosons interacting only with external fields or sources, the unquantized fields continue to satisfy the deterministic equations of motion of the unquantized theory. It seems significant that classical relativistic fields with external sources are not eliminated by the process of quantization. All the observable predictions of quantum theory are identically reproduced and no new results are obtained. This method of quantization is therefore an alternative to the usual one. A natural extension of the theory yields a deterministic model for direct boson-boson interactions (*e.g.* pion electrodynamics), free of mathematical divergences. This will be developed in a following paper.

1. – Introduction.

While a future physical theory will have to yield the verified predictions of quantum mechanics, it may present these in a new mathematical form. This possibility makes it worthwhile to search for alternative methods of quantization.

The aim of this paper is to introduce a new method of quantization which is an alternative to the usual one. Both methods are of comparable mathe-

matical simplicity and lead to the same observable predictions. They are not however simply equivalent in the sense that, say, matrix mechanics is equivalent to wave mechanics. The new method leads, for instance, to the introduction of additional physical observables, in particular classical canonical variables for those systems possessing a classical analog.

The usual method of canonical quantization eliminates the canonical variables of a classical theory by replacing them with observables which satisfy uncertainty relations. As a result, the quantized theory is indeterministic. This situation has not unnaturally led to the impression that the existence of complementary (*i.e.* not simultaneously measurable) observables, by itself contradicts the possibility of a deterministic physical theory (we shall call a theory deterministic when *a*) the statistical state of the system it describes can be completely specified by the joint distribution of certain parameters and *b*) these parameters satisfy an equation of motion of first order in the time. An example is classical statistical mechanics; the parameters in this case are the canonical variables).

Bohm [1] has proposed a deterministic re-interpretation of quantum theory in which, accordingly, complementary observables play no part. Bohm's theory is therefore classical in the usual sense. Quantum effects are accounted for by special « quantum forces », of random character, introduced for this purpose. As we shall see, however, it is not in fact necessary in order that a physical theory be deterministic, that it should exclude complementary observables. Thus, even if one were to adopt the arbitrary philosophical position that the ultimate goal of physics is the construction of a completely objective model of reality [2], there would be no reason for ruling out complementary observables.

Much confusion has grown around this question because of the difficulty of interpreting a theorem due to von Neumann [3], concerning the possible role of hidden parameters in physics. Contrary to what seems to be generally assumed, this theorem does not disprove the possibility of the deterministic description of a physical system possessing complementary observables.

The theorem assumes in effect that a physical system must indeed own some complementary observables and goes on to draw the conclusion that if hidden parameters exist, not all observables can be functions of these parameters: since, even if the values of all hidden parameters are given, not all observables can be simultaneously known. However the theorem does not

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[1] D. Bohm: *Phys. Rev.*, **85**, 166 (1952); D. Bohm and J. P. Vigier: *Phys. Rev.*, **96**, 208 (1954).

[2] P. A. Schilpp: *Albert Einstein: Philosopher-Scientist* (New York, 1951), p. 675.

[3] J. V. Neumann: *Mathematische Grundlagen der Quantenmechanik* (New York, 1943), pp. 170-173.

prove (it does not actually claim to) that the state (*i.e.* the total information required to determine the distribution of every observable) of the system cannot be completely specified by the values of the hidden parameters or, in general, by their joint distribution (4).

There exist in fact several familiar examples of deterministic theories which are also capable of describing complementary observables. To consider only one: the pitch ν and the epoch t of a musical note may be classed as complementary acoustical observables since they satisfy the relation (5)

$$\Delta\nu\,\Delta t \geqslant 1/4\pi\,.$$

And yet acoustical theory is deterministic.

Thus, contrary to a commonly held impression, a deterministic model for a physical system is not *a priori* inconsistent with BOHR's fundamental idea of a complementary description of physical observables. While this conclusion in no way suggests that a convincingly simple deterministic model of all physical phenomena can be found, it does imply that the physical world may be indeterministic only in part. This is certainly possible at least if one is willing to describe the world as composed of separate parts, *i.e.* if he does not insist on a unified theory. If we agree to adopt this viewpoint, at least temporarily, we are then naturally led to reconsider the reasons for giving up the « successful » (relativistic) classical models, namely the classical field models of gravitation and radiation in the presence of external sources.

The fact that electromagnetic energy is quantized implies that the continuous classical expression $\frac{1}{2}\int d\tau(e^2+h^2)$ cannot be interpreted as energy. It was essentially the difficulty of finding a quantized expression for electromagnetic energy within a theoretical framework which would also include the classical field that led PLANCK finally to abandon his attempt to preserve this field and DIRAC to replace it by its quantized counterpart (6). Although this method of quantization was completely successful, the foregoing considerations make it worthwhile to reconsider the question of the necessity of giving up the classical field. The problem may be stated as follows: « Can an alternative method of quantization be found which does not destroy the classical field? » If this is possible, the quantized and classical fields may clearly not be identified with each other: for instance, the components of the former are joined by the Heisenberg uncertainty relations whereas those of the latter are not. Once this necessary distinction between the two fields is realized, certain paradoxes are more easily avoided.

(4) F. BOPP: *Ann. Inst. Henri Poincaré*, **15**, no. 2, 81 (1956).

(5) $\Delta\nu$, Δt are not in this case statistical dispersions; they simply give the accuracy to which ν and t are defined.

(6) P. A. M. DIRAC: *Proc. Roy. Soc.*, **114**, 243*a*, 710 (1927).

In the following Section a new general method of quantization is presented. When applied to a system with a classical analog, this method introduces quantized canonical variables without however eliminating the corresponding unquantized variables: their (non-negative definite) distribution in phase space then determines the state of the system. The physical meaning of these classical (unquantized) variables is made clear by the proof that they always have the same expectation value as, and no greater dispersion than, their quantized counterparts. The measurement of a quantized variable thus produces a value for the corresponding classical observable.

Classical boson fields in particular are retained as legitimate physical observables. When bosons interact only with external fields or sources, the classical boson fields are shown to satisfy the deterministic equations of motion of the unquantized theory. It is worth noting especially that the classical electromagnetic and gravitational (and also mesic) fields with external sources are not eliminated by the new process of quantization ([6,7]). All the predictions of quantum theory are identically verified and no new results are obtained.

2. – Quantization.

Our starting point is the Hilbert space of quantum theory.

2·1. *States.* – We postulate that the state of a system is represented by a time-dependent self-adjoint linear operator Φ. This Φ is not the statistical operator U of VON NEUMANN ([8]), but we shall find later that

$$U = \Phi^2 .$$

If M, N are two linear operators, we define their scalar product (M, N) by the relation

$$(M, N) = \mathrm{Tr}\,(M^\dagger N) .$$

It follows that, if (M, N) exists,

$$(M, N) = (N, M)^* ,$$

and if (M, M) exists (M is then said to be normalizable),

$$(M, M) \geqslant 0 .$$

([7]) S. N. GUPTA: *Rev. Mod. Phys.*, **29**, 334 (1957).

([8]) J. V. NEUMANN: *Mathematische Grundlagen der Quantenmechanik* (New York, 1943) p. 168.

Normalizable linear operators form a manifold which is itself a Hilbert space. We shall refer to it as the *associated Hilbert space* (AHS).

2·2. *Observables.* – Physical observables will be represented by self-adjoint linear *superoperators* defined as follows: a superoperator is an operator in AHS. The theory of operators in Hilbert space [9] is therefore directly applicable to superoperators. For example, a superoperator s transforms every linear operator M of its domain in AHS into some normalizable linear operator M':

$$sM = M' .$$

Superoperators s, $s^\dagger$ are said to be adjoint if they have the same domain, and in that domain

$$(s^\dagger M,\ N) = (M,\ sN) .$$

The linear character of s means that its domain is a linear manifold and, if $a, ..., d$ are complex numbers,

$$s(aM + ... + dP) = asM + ... + dsP .$$

2·3. *Expectation values.* – It is postulated that the expectation value of the observable s in the state Φ is [10]

$$\langle s \rangle = (\Phi,\ s\Phi) .$$

This requires

$$(\Phi,\ \Phi) = 1 . \tag{1}$$

It is further assumed that the observable $f(s)$, where $f(s)$ is a real power series in x, is represented by the superoperator $f(s)$, and the observable $s+t$ by the superoperator $s+t$.

If Φ is multiplied by an arbitrary phase factor, expectation values are evidently unchanged. However the requirement that Φ is self-adjoint restricts this factor to ± 1. Thus $+\Phi$ and $-\Phi$ represent the same physical state.

2·4. *Special observables.*

a) Twin quantized observables. – Let A be a self-adjoint linear operator in Hilbert space ($A = A^\dagger$). Then the superoperator a' will be de-

[9] J. V. Neumann: *Mathematische Grundlagen der Quantenmechanik* (New York, 1943) p. 47-48.

[10] For convenience, we use the same symbol s for the observable as for its associated superoperator.

fined by [11]

$$a^l M = AM\,, \qquad \text{for every } M. \tag{2}$$

a^l is obviously linear, and we have

$$(a^l M, N) = \mathrm{Tr}\,((AM)^\dagger N) = \mathrm{Tr}\,(M^\dagger A N) = (M, a^l N)\,.$$

Thus a^l is self-adjoint. We find

$$\langle f(a^l)\rangle = (\Phi, f(a^l)\Phi) = \mathrm{Tr}\,(\Phi\ (A)\Phi) = \mathrm{Tr}\,(U f(A))\,, \tag{3}$$

where

$$U = \Phi^2\,.$$

We may also define [11]

$$a^r M = MA\,, \qquad \text{for every } M. \tag{4}$$

a^r is also linear and self-adjoint, and we find

$$\langle f(a^r) = \mathrm{Tr}\,(U f(A)) = \langle f(a^l)\rangle\,. \tag{5}$$

Thus the observables represented by a^l and a^r are *indistinguishable*, in essentially the same sense already familiar from the quantum theory of identical particles (*e.g.* the co-ordinates x_1, x_2 of two electrons are indistinguishable observables in quantum theory).

If U is now identfied with the statistical operator of von Neumann, the expression

$$\mathrm{Tr}\,(U f(A))$$

is the expectation value of the *quantum-mechanical* observable $f(A)$. It is clear then that, according to the theory proposed here, what is called in quantum mechanics the observable A actually represents two indistinguishable « twin » observables a^l and a^r. This doubling of all quantum observables suggests that this method of quantization might be called *double quantization*.

When a^l (or a^r) is precisely known, *i.e.*

$$\langle a^{l^2}\rangle - \langle a^l\rangle^2 = 0\,,$$

we have, from (3)

$$\mathrm{Tr}\,(UA^2) - [\mathrm{Tr}\,(UA)]^2 = 0\,.$$

---

[11] This definition will hold also when A is not self-adjoint.

This is well known to imply that Tr (UA) is an eigenvalue of the operator A. Thus, from (3), when a^l is precisely known, its value is an eigenvalue of A.

b) Classical observables. – It is of interest to examine the properties of the observable

$$a = \tfrac{1}{2}(a^l + a^r) . \tag{6}$$

A real function $f(a)$ of such an observable, and also a sum

$$f(a) + g(b) + \ldots$$

of such functions, may conveniently be called a *classical observable*. Classical observables do not necessarily commute with each other, but we shall see in the next Section that classical canonical variables, defined according to (6), do mutually commute.

It follows from (3), (5) that

$$\langle a \rangle = \langle a^l \rangle = \langle a^r \rangle = \mathrm{Tr}\,(UA) .$$

Consider now the dispersion of a. We have

$$\begin{aligned} \langle (\Delta a)^2 \rangle &= \langle a^2 \rangle - \langle a \rangle^2 \\ &= \tfrac{1}{4}[\langle a^{l^2} \rangle + \langle a^l a^r \rangle + \langle a^r a^l \rangle + \langle a^{r^2} \rangle] - \langle a^l \rangle^2 \\ &= \tfrac{1}{2}\langle a^{l^2} \rangle + \tfrac{1}{4}\langle a^l a^r \rangle + \tfrac{1}{4}\langle a^r a^l \rangle - \langle a^l \rangle^2 . \end{aligned}$$

It is readily verified that « l » observables commute with « r » observables:

$$[a^l, b^r]M = A(MB) - (AM)B = 0 , \qquad \text{for every } M.$$

Thus

$$[a^l, b^r] = 0 .$$

In particular $[a^l, a^r] = 0$. We can then define a joint probability distribution $P(a^l, a^r)$ as follows:

$$P(x, y) = \langle \delta(a^l - x)\, \delta(a^r - y) \rangle ,$$

where

$$\delta(a^l - x) = (1/2\pi) \int_{-\infty}^{\infty} \mathrm{d}z \, \exp\,[iz(a^l - x)] , \quad \text{etc.} ,$$

and $\exp[iz(a^l - x)]$ may be written as a power series in $iz(a^l - x)$. It is easily proved that this distribution is non-negative definite. We can then use the Schwarz inequality

$$\langle a^l a^r \rangle = \langle a^r a^l \rangle \leqslant \langle a^{l2} \rangle^{\frac{1}{2}} \langle a^{r2} \rangle^{\frac{1}{2}} = \langle a^{l2} \rangle .$$

We then obtain

$$\langle (\Delta a)^2 \rangle \leqslant \langle a^{l2} \rangle - \langle a^l \rangle^2 = \langle (\Delta a^l)^2 \rangle = \langle (\Delta a^r)^2 \rangle .$$

Thus a has the same expectation value as, and no greater dispersion than, a^l (or a^r). The measurement of the « quantized » observable a^l (or a^r) (which in quantum theory is interpreted as a measurement of A) produces therefore a value of the classical observable a with at least as good accuracy.

It can also be shown that the uncertainty relations connecting any two classical observables σ, τ take the trivial form

$$\langle (\Delta\sigma)^2 \rangle^{\frac{1}{2}} \langle (\Delta\tau)^2 \rangle^{\frac{1}{2}} \geqslant 0 .$$

2·5. *Equation of motion.* – The mathematical apparatus presented above closely resembles that of quantum theory, the state operator Φ playing the role of the pure state vector in quantum mechanics. Pursuing this analogy we write down the equation of motion as follows

$$i\hbar\, \partial\Phi/\partial t = k\Phi , \tag{7}$$

where k is a self-adjoint superoperator which we shall call the superhamiltonian of the system. This guarantees that the normalization condition (1) is conserved. k must also be so chosen that Φ remains self-adjoint. The time-derivative $\dot{s}$ of an observable s may then be defined by the equation

$$i\hbar\dot{s} = i\hbar\, \partial s/\partial t + [s, k] .$$

In this paper we shall limit ourselves to a choice of k which guarantees that the results of quantum theory are identically reproduced. We choose

$$k = h^l - h^r , \tag{8}$$

H being the quantum-mechanical Hamiltonian ordinarily associated with the system. Thus k does not represent energy. To see that this choice of k does in fact lead to the results of quantum theory, we note that (7) may be written

$$i\hbar\, \partial\Phi/\partial t = H\Phi - \Phi H ,$$

which conserves automatically the self-adjoint character of Φ. Von Neumann's equation of motion ([12])

$$i\hbar\, \partial U/\partial t = HU - UH$$

immediately follows.

Later (Sect. **6**) we shall have occasion to consider the possibility of choosing k differently.

3. – Classical canonical variables.

Consider the linear operators Q, P satisfying

$$[Q, P] = i\hbar\,.$$

It follows readily from (2) and (4) that

$$[q^l, p^l] = \quad i\hbar\,,$$

$$[q^r, p^r] = -i\hbar\,.$$

If we define the classical canonical variables according to (6):

$$q = \tfrac{1}{2}(q^l + q^r)\,,$$

$$p = \tfrac{1}{2}(p^l + p^r)\,,$$

we find

$$[p, q] = 0\,.$$

In general, if we have several linear operators, *i.e.*

$$[Q_i, P_j] = i\hbar\,\delta_{ij}\,,$$

we find that the set of classical variables q, p mutually commute.

If we define the linear expressions

$$(9) \qquad \begin{cases} A = \sum_i c_i Q_i + d_i P_i\,, \\ B = \sum_i e_i Q_i + f_i P_i\,, \end{cases}$$

(c, d, e, f complex numbers), it follows that

$$(10) \qquad [a, b] = 0\,.$$

([12]) J. V. NEUMANN: *Mathematische Grundlagen der Quantenmechanik* (New York, 1943) p. 186.

4. – Quantum electrodynamics.

We shall now apply these considerations to quantum electrodynamics. The Maxwell potentials A_μ in quantum theory may be assumed to satisfy the equations

$$\Box A_\mu = -(1/c)J_\mu . \tag{11}$$

It follows that

$$\Box a^l_\mu = -(1/c)j^l_\mu ; \qquad \Box a^r_\mu = -(1/c)j^r_\mu .$$

and therefore

$$\Box a_\mu = -(1/c)j_\mu .$$

The usual quantized field components

$$F_{\mu\nu} = \partial A_\nu/\partial x_\mu - \partial A_\mu/\partial x_\nu$$

are expressions of the type (9), and therefore the classical Maxwell field components

$$f_{\mu\nu} = \partial a_\nu/\partial x_\mu - \partial a_\mu/\partial x_\nu$$

mutually commute, according to (10). According to the discussion of classical observables in Sect. 2·4, a measurement of $F_{\mu\nu}$ produces a value for $f_{\mu\nu}$.

If we neglect the sources, the equation

$$\Box a_\mu = 0$$

shows clearly that the propagation of the classical potentials is deterministic. The same is true if the sources are external, so that j^l_μ, j^r_μ and j_μ are replaced by the same explicitly given function of the space-time co-ordinates.

The supplementary condition needed to insure that $f^l_{\mu\nu}$, $f^r_{\mu\nu}$ obey Maxwell's equations may be written, according to Fermi's method [13]

$$\chi\Phi = \Phi\chi = 0 ,$$
$$\dot\chi\Phi = \Phi\dot\chi = 0 ; \qquad \chi = \partial A_\nu/\partial x_\nu .$$

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[13] G. WENTZEL: *Quantum theory of fields* (New York, 1949), p. 112.

These conditions are conserved by the equation (11), and

$$\partial J_r / \partial x_r = 0 .$$

5. – Boson fields.

If we neglect all interactions except those with external sources or fields, the quantum-mechanical Hamiltonian $H(Q, P)$ of a boson field is a quadratic function of the Q_i, P_i, the coefficients being possibly time-dependent (see equation (19)). The canonical equations of motion

$$\dot{Q}_i = \partial H(Q, P)/\partial P_i ; \qquad \dot{P}_i = - \partial H(Q, P)/\partial Q_i$$

are then linear. It follows from (2), (4) and (6) that the « l » and « r » canonical variables as well as the classical canonical variables satisfy the same equation. We have

$$\dot{q}_i = \partial H(q, p)/\partial p_i ; \qquad \dot{p}_i = - \partial H(q, p)/\partial q_i . \tag{12}$$

Thus the classical canonical variables obey the deterministic equations of motion of the unquantized theory. It follows that the probability distribution $g(q, p; t)$ in classical phase space must satisfy the Liouville equation ([14]). In the following Section we shall construct an expression for $g(q, p; t)$ for an arbitrary physical system and verify the Liouville equation for the case discussed in the present Section. We will at the same time show, for this case, that the theory of bosons is deterministic and furnishes a model identical with the one provided by the unquantized theory. This will enable us to formulate a suggestion for a new treatment of direct boson-boson interactions.

6. – General canonical representation.

The state operator Φ of an arbitrary physical system may be represented as a matrix if an orthogonal basis in Hilbert space is chosen. More generally we shall consider a Dirac representation based upon the usual canonical coordinates Q_i possessing a classical analog and other variables N_s without classical analog (for instance fermion occupation numbers). If a boson field ψ

([14]) For charged boson fields, the canonical variables as usually defined are complex. They may however be replaced by real variables (cfr. beginning of the following Section).

is complex it may be decomposed into real fields ψ_1, ψ_2 according to the relations ([15])

$$\psi = (1/\sqrt{2})(\psi_1 + i\psi_2)$$
$$\psi^\dagger = (1/\sqrt{2})(\psi_1 - i\psi_2)$$

and we may therefore assume without loss of generality that the Q_i are real. The « matrix element » of Φ is then

$$\langle Q' N' | \Phi | Q'' N'' \rangle \tag{13}$$

in the usual Dirac notation.

We shall not for the moment make use of the fact that Φ is self-adjoint, and for brevity we omit reference to N', N'' in the matrix element, writing $\langle Q' | \Phi | Q'' \rangle$ in place of (13).

In order to represent the superoperators q_j^l, q_j^r, p_j^l, p_j^r we consider for instance the matrix element

$$\langle Q' | q_j^l \Phi | Q'' \rangle = \langle Q' | Q_j \Phi | Q'' \rangle = \int \mathrm{d}Q''' \langle Q' | Q_j | Q''' \rangle \langle Q''' | \Phi | Q'' \rangle = Q_j' \langle Q' | \Phi | Q'' \rangle .$$

Similarly we find

$$\langle Q' | q_j^r \Phi | Q'' \rangle = Q_j'' \langle Q' | \Phi | Q'' \rangle .$$

If the operator P_j in this representation is given by

$$P_{j_{op}} = (\hbar/i)\, \partial/\partial Q_j$$

we obtain

$$\langle Q' | p_j^l \Phi | Q'' \rangle = (\hbar/i)\, \partial \langle Q' | \Phi | Q'' \rangle / \partial Q_j' .$$

Likewise,

$$\langle Q' | p_j^r \Phi | Q'' \rangle = \int \mathrm{d}Q''' \langle Q' | \Phi | Q''' \rangle \langle Q''' | P_j | Q'' \rangle = \int \mathrm{d}Q''' \langle Q'' | P_j | Q''' \rangle^* \langle Q' | \Phi | Q''' \rangle =$$
$$= -(\hbar/i)\, \partial \langle Q' | \Phi | Q'' \rangle / \partial Q_j'' .$$

We see that the superoperators q_j^l, q_j^r, p_j^l, p_j^r are represented by the following hermitian functional operators acting upon the matrix element $\langle Q' | \Phi | Q'' \rangle$

([15]) G. WENTZEL: *Quantum theory of fields* (New York, 1949), p. 12.

considered as a function of the variables Q_i', Q_i'', t:

$$(14)\qquad \begin{cases} (q_j^l)_{op} = Q_j' ; & (q_j^r)_{op} = Q_j'' \\ (p_j^l)_{op} = (\hbar/i)\,\partial/\partial Q_j' ; & (p_j^r)_{op} = -(\hbar/i)\,\partial/\partial Q_j'' . \end{cases}$$

For instance $\langle Q' | p_j^l \Phi | Q'' \rangle$ is a function of Q_i', Q_i'', t which is obtained by operating upon $\langle Q' | \Phi | Q'' \rangle$ with the hermitian operator $(\hbar/i)\,\partial/\partial Q_j'$.

Starting from this representation we may then make use of the technique of transformation theory to obtain other representations. We consider first a simple change of variables:

$$(15)\qquad \begin{cases} Q_j' = q_j' + \tfrac{1}{2} k_j' , \\ Q_j'' = q_j' - \tfrac{1}{2} k_j' . \end{cases}$$

We write

$$\langle Q' | \Phi | Q'' \rangle = \chi(q', k'; t) .$$

From (15) follows

$$(16)\qquad \begin{cases} (\hbar/i)\,\partial/\partial Q_j' = (\hbar/i)\,\partial/\partial k_j' + (\hbar/2i)\,\partial/\partial q_j' , \\ -(\hbar/i)\,\partial/\partial Q_j'' = (\hbar/i)\,\partial/\partial k_j' - (\hbar/2i)\,\partial/\partial q_j' . \end{cases}$$

Recalling (14), the relations (15) and (16) give us the hermitian operators representing q_j^l, q_j^r, p_j^l, p_j^r in the q', k' representation. We can now very simply obtain a canonical representation by performing the Fourier transformation

$$(17)\qquad \Psi(q', p'; t) = \int \cdots \int \prod_i \frac{\mathrm{d}k_i'}{\sqrt{2\pi\hbar}}\, \chi(q', k'; t) \exp\left[-i \prod_i \frac{k_i' p_i'}{\hbar} \right].$$

This yields

$$p_j' \Psi(q', p'; t) = \int \cdots \int \prod_i \frac{\mathrm{d}k_i'}{\sqrt{2\pi\hbar}}\, [(\hbar/i)\,\partial\chi(q', k'; t)/\partial k_j'] \exp\left[-i \prod_i \frac{k_i' p_i'}{\hbar} \right]$$

$$i\hbar\,\partial\Psi(q', p'; t)/\partial p_j' = \int \cdots \int \prod_i \frac{\mathrm{d}k_i'}{\sqrt{2\pi\hbar}}\, [k_j' \chi(q', k'; t)] \exp\left[-i \prod_i \frac{k_i' p_i'}{\hbar} \right].$$

Thus, in the canonical representation, q_j^l, q_j^r, p_j^l, p_j^r are given by

$$(18)\qquad \begin{cases} (q_j^l)_{op} = q_j' + (i\hbar/2)\,\partial/\partial p_j' ; & (q_j^r)_{op} = q_j' - (i\hbar/2)\,\partial/\partial p_j' , \\ (p_j^l)_{op} = p_j' - (i\hbar/2)\,\partial/\partial q_j' ; & (p_j^r)_{op} = p_j' + (i\hbar/2)\,\partial/\partial q_j' . \end{cases}$$

It follows in particular that

$$(q_j)_{op} = q_j' ,$$
$$(p_j)_{op} = p_j' .$$

The probability distribution function $g(q', p'; t)$ in the classical phase space may now be obtained. Let $f(q, p)$ be a function of the canonical variables expressible as a power series. We obtain

$$\langle f(q,p)\rangle = \sum_{N_s'}\sum_{N_s''}\int\!\cdots\!\int\prod_i \mathrm{d}Q_i'\,\mathrm{d}Q_i''\,\langle Q'N'|\Phi|Q''N''\rangle^*\cdot\langle Q'N'|f(q,p)\,\Phi|Q''N''\rangle =$$

$$= \sum_{N_s'}\sum_{N_s''}\int\!\cdots\!\int\prod_i \mathrm{d}q_i'\,\mathrm{d}p_i'\,f(q',p')\,|\Psi(q',p';N',N'';t)|^2 =$$

$$= \int\!\cdots\!\int\prod_i \mathrm{d}q_i'\,\mathrm{d}p_i'\,g(q',p';t)f(q',p')\,.$$

where we have written in the explicit dependence on N_s', N_s''.

Thus the (non-negative definite) classical phase space distribution is (dropping primes)

$$g(q,p;t) = \sum_{N_s'}\sum_{N_s''}|\Psi(q,p;N',N'';t)|^2\,.$$

It may readily be verified that the requirement that Φ is self-adjoint,

$$\langle Q'N'|\Phi|Q''N''\rangle = \langle Q''N''|\Phi|Q'N'\rangle^*\,,$$

is equivalent to the condition

$$\Psi(q,p;N',N'';t) = \Psi^*(q,p;N'',N';t)\,.$$

In particular, if the physical system has a classical analog, the variables N_s', N_s'' do not occur and Ψ is real. We have then

$$g = \Psi^2\,.$$

If in that case g is a given analytic function of the classical canonical variables q_i, p_i, its square root Ψ is given by

$$\Psi = \pm\sqrt{g}\,, \quad \text{analytic},$$

i.e. is determined except for the sign. Inversion of the transformations (17) and (15) then determines Φ except for a factor ± 1. We have seen however (Section 2·3) that two Φ's differing in sign determine the same physical state. Thus, whenever the system has a classical analog, the classical phase space distribution (assumed to be analytic) determines the state of the system. When,

in addition, the q_i, p_i satisfy first order equations of motion, as in (12), the theory of the system is deterministic.

In this case (bosons interacting only with external sources or fields), the quantum-mechanical Hamiltonian is a quadratic expression

$$(19) \qquad H(Q, P) = \sum_{ij} a_{ij} P_i P_j + b_{ij} Q_i Q_j + \tfrac{1}{2} c_{ij} (P_i Q_j + Q_j P_i) + \sum_i a_i P_i + b_i Q_i + c \,.$$

The equation of motion for Ψ may then be obtained by writing (7) in the canonical representation. It is readily verified that

$$h^l = H(q^l, p^l) \,; \qquad h^r = H(q^r, p^r) \,.$$

Then

$$k = H(q^l, p^l) - H(q^r, p^r) \,.$$

If the operators (18) are substituted for q_i^l, q_i^r, p_i^l, p_i^r in this expression, we find after some cancellations:

$$(20) \qquad k_{op} = i\hbar \sum_i \frac{\partial H(q', p')}{\partial q_i'} \frac{\partial}{\partial p_i'} - \frac{\partial H(q', p')}{\partial p_i'} \frac{\partial}{\partial q_i'} \,.$$

The equation of motion

$$i\hbar \, \partial \Psi / \partial t = k_{op} \Psi$$

is then simply the Liouville equation for Ψ. Since it is linear in the differential operators $\partial/\partial q_i'$, $\partial/\partial p_i'$, $\partial/\partial t$, it follows that $g = \Psi^2$ also satisfies this equation ([16]), as is required, since in this case the classical canonical variables obey the classical equations of motion (12).

The canonical representation is of some interest as it suggests a possible way of modifying the physical theory.

The fact that the deterministic equations (12) are satisfied when bosons interact only with external fields or sources, strongly suggests that the theory should be so modified that they remain satisfied also when bosons interact directly with each other. Ψ would then, also in that case, obey the Liouville equation ([17]). If this were indeed the case, Ψ would remain real, and Φ therefore self-adjoint as required. Also the superhamiltonian would have the form (20) which, as one can easily verify, guarantees that it is self-adjoint. However k

([16]) See G. A. Baker jr.: *Phys. Rev.*, **109**, 2198 (1958), Appendix.

([17]) For instance in the electrodynamics of charged pions (the electrodynamics of deuterons is of course not a direct boson-boson interaction).

would no longer separate the « l » and « r » variables as in (8), and departures from the predictions of quantum theory would be met.

This program will be carried out for charged boson electrodynamics in a following paper. This deterministic model is also interesting since, being a classical field model, it is manifestly free of mathematical divergences.

* * *

The author would like to thank Profs. G. CHEW and P. SWEET for very helpful discussions.

RIASSUNTO (*)

Si presenta un nuovo metodo di quantizzazione applicabile ad un sistema fisico arbitrario. Per sistemi aventi un analogo classico l'introduzione di variabili canoniche quantizzate si esegue senza eliminare le corrispondenti variabili non quantizzate (benchè l'equazione di Liouville sia in genere modificata). Queste restano come osservabili fisiche legittime avendo lo stesso valore probabile e dispersione non maggiore dei loro equivalenti quantizzati. La loro distribuzione (non negativa, definita) nello spazio delle fasi determina allora lo stato del sistema. Nel caso di bosoni interagenti solo con campi o sorgenti esterni i campi non quantizzati continuano a soddisfare le equazioni deterministiche di moto della teoria non quantizzata. Appare significativo che campi relativistici con sorgenti esterne non siano eliminati dal processo di quantizzazione. Tutte le predizioni osservabili della teoria quantistica si riproducono identicamente e non si ottengono nuovi risultati. Il presente metodo di quantizzazione è pertanto un'alternativa per quello usuale. Una naturale estensione della teoria fornisce un modello deterministico per le interazioni dirette bosone-bosone (p.e. l'elettrodinamica dei pioni) esente da divergenze matematiche. Questo sarà sviluppato in un prossimo lavoro.

(*) *Traduzione a cura della Redazione.*

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Magnetic Perturbation of Cathode Rays.

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(ricevuto il 29 Agosto 1958)

Summary. — When an electron beam penetrates a magnetized body, splitting of the electron beam occurs. This phenomenon is explained in terms of polarization of the cathode rays.

1. – Introduction.

An electron beam is perturbed by magnetic fields, and spreading and deflection of the beam takes place here. The behavior of an electron beam that penetrates a magnet was studied in the present work. Splitting of the beam, distinguishable from the above-mentioned phenomena, was observed after the beam had tunneled through the magnet body. The author has elucidated this phenomenon as the polarization of free electrons.

2. – Experimental.

The present experiment was carried out with an electron diffraction apparatus. A single crystal of iron (length: 10 mm, diameter: 1 mm) was edged mechanically and then the edge thus prepared was etched with hydrochloric acid. One piece of permanent magnet (coercive force, about 700 Oe) was attracted to the rear side of the test piece, as illustrated in Fig. 1. In this way the single crystal was magnetized. An electron beam (wavelength, 0.0308 Å) grazed the edge of the single crystal. The diffraction pattern so obtained is shown in Fig. 2. In Fig. 2 the incident beam runs nearly perpendicular to the (111) plane of

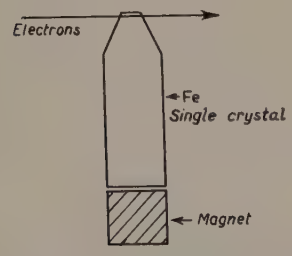

Fig. 1. – Arrangement of experiment.

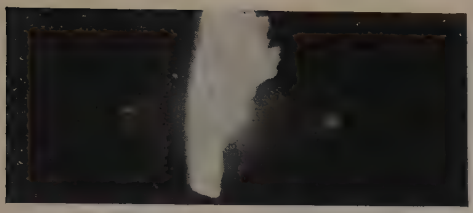

Fig. 2. – Diffraction pattern obtained from the magnetized single crystal of iron. Wavelength, 0.0308 Å. Camera length, 495 mm. Positive enlarged 2.3 times.

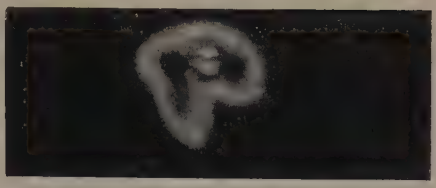

Fig. 3. – The central spot found in Fig. 2, enlarged optically 60 times. One sees splitting of the electron beam.

the iron crystal. The central spot found in Fig. 2 was enlarged optically 60 times, as shown in Fig. 3. It should be noticed in Fig. 3 that the electron beam is split into discrete beams. Fig. 4 shows the incident beam which grazed the non-magnetized test piece. There is no splitting of the beam.

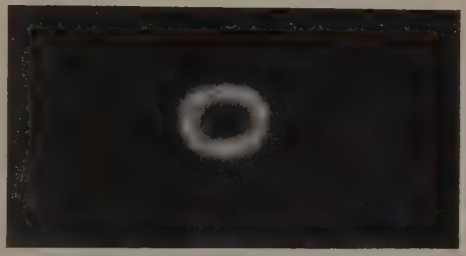

Fig. 4. – The electron beam which grazed the non-magnetized specimen. 60 times enlarged. There is no splitting.

3. – Discussion I.

When the electrons pass through a magnetized body, interaction occurs between the spins of the incident electrons and the steep gradient of the magnetic field. This gradient of the field exists at the border between two Weiss magnetic domains and the Bloch magnetic wall. In Fig. 5, the relation between the direction of incidence and the field is illustrated. The deflection ΔZ of the spinning electrons by the magnetic field is given by the equation

$$\Delta Z = B \left| \frac{\partial H}{\partial Z} \right| m \left(\frac{\lambda}{h} \right)^2 Ll , \qquad (L \gg l), \tag{1}$$

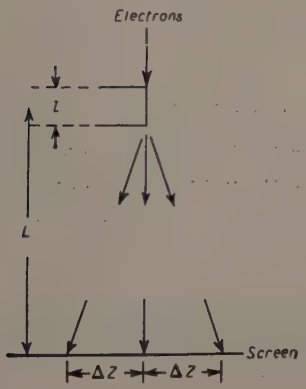

Fig. 5. – Relation between the incident electrons and the magnetic field. L, camera length. l, dimension of Weiss domain. ΔZ, observed splitting.

where B: the Bohr magneton ($0.93 \cdot 10^{-20}$ erg/gauss), $\partial H/\partial Z$: the gradient of the magnetic field, m: electron mass ($9.1 \cdot 10^{-28}$ g), λ: the wavelength of the electrons (0.030 8 Å), l: the distance travelled by the electrons in the region of magnetic field gradient, L: the camera length (495 mm), and h: Planck's constant ($6.6 \cdot 10^{-27}$ erg·s).

The spacing measured between the split spots in Fig. 3 (ΔZ) is 0.01 cm. The linear dimension of the Weiss domain (10^{-3} cm) was employed as the value of l in Eq. (1). Therefore, from Eq. (1) we obtain

$$\left| \frac{\partial H}{\partial Z} \right| \approx 10^{13} \text{ G/cm} .$$

It is expected that there is a steep gradient of the magnetic field at the border between the Weiss domains and the Bloch wall, especially at the surface of the magnet body. It is known that the thickness ($\Delta\zeta$) of the Bloch wall is about 100 Å. If we assume

$$\left| \overline{\frac{\partial H}{\partial Z}} \right| \approx \frac{\Delta H}{\Delta \zeta} ,$$

we obtain

$$\Delta H \simeq 10^7 \text{ G} ,$$

where ΔH means the strength of the Weiss magnetic field. This observed value is approximately equal to the known value of the Weiss field.

Here the author has considered the Weiss field as a real magnetic field. This treatment is not quantal, but classical. This is quite similar to Bragg's relation regarding net planes in electron diffraction, which regards interplanar spacing as real. In quantal discussion, the present electron polarization should be treated as the effect of the exchange force between the spins of the incident electrons and the oriented spins in the magnet.

4. – Discussion II.

According to the uncertainty principle, it is impossible to polarize a beam of free electrons into two beams of separate spins by means of the Stern-Gerlach process (1). Spreading of a flux of electrons by a magnetic field with gradient masks completely the splitting of the electron spins by the same field. The

(1) N. F. MOTT: *Proc. Roy. Soc.*, A **124**, 440 (1929); N. F. MOTT and H. S. W. MASSEY: *The theory of atomic collisions*, 2nd ed. (Oxford, 1950), p. 61; E. VAN DER SPUY: *Nuovo Cimento*, **4**, 1349 (1956).

force required to split the spinning electrons (F_s) is negligible as compared with that to spread the electron beam (F_b). That is

$$F_b \gg F_s .$$

This is because

$$\Delta x \cdot v \gg \frac{h}{m},$$

or

$$\Delta x \gg \lambda ,$$

where v is the velocity of the free electrons, Δx is the cross-section of the electron flux, and λ is the wavelength of the free electrons. It is impossible to observe the electron spins in the Stern-Gerlach process, insofar as Δx, that is, a slit for the electron beam, is not comparable with the wavelength applied. Here, the intensity of the electron beam passing through this narrow slit is not only too weak to be observed, but also it is perturbed physical-optically at the slit.

In the present work, it is shown that there is a natural slit of the order of the de Broglie wavelength, whose periphery does not screen the incident beam. This slit does never perturb the incident beam physical-optically. There is a very steep gradient of the magnetic field at this slit, *i.e.* at the border between the Weiss domains and the Bloch wall. On grazing the surface of a magnet body, the electron beam is perturbed by the steep gradient of the field.

RIASSUNTO (*)

Quando un elettrone penetra in un corpo magnetizzato si verifica una separazione del fascio di elettroni. Si spiega tale fenomeno in termini della polarizzazione dei raggi catodici.

(*) *Traduzione a cura della Redazione.*

The Use of Methyl Iodide-Propane Mixtures and Tri-Fluoro Bromo Methane as Bubble Chamber Fluids.

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(ricevuto il 1° Agosto 1958)

Summary. — The suitability of a methyl iodide-propane system as a « heavy liquid » bubble chamber medium has been investigated. Operating conditions have been found for various mixtures, and also for tri-fluoro bromo methane, another possible heavy liquid medium.

1. – Introduction.

Attempts have been made in various laboratories ([1,3]) to find a bubble chamber medium having high density and atomic number, and therefore good γ conversion properties. One of the most promising liquids from the point of view of radiation length alone, would appear to be CH_3I, but unfortunately iodine is liberated from this material under the influence of light and heat, and the liquid rapidly becomes photographically unsuitable. Several substances have been suggested as possible inhibitors of the disintegration, the most popular of which is propane since it is itself a well-tried bubble chamber liquid of fairly low operating temperature, and therefore may be expected to lower the inconveniently high operating temperature (210 °C) of pure CH_3I.

2. – Chamber.

We have constructed a small chamber $1\frac{1}{2}$ in. dia. $\times 1$ in. entirely of monel metal in which heavy liquids can be investigated. Expansion is by means

([1]) J. L. Brown, D. A. Glaser and M. L. Perl: *Phys. Rev.*, **102**, 586 (1956).
([2]) E. D. Alyea jr., L. R. Gallagher, J. H. Mullins and J. M. Teem: *Nuovo Cimento*, **6**, 1480 (1957).
([3]) I. Pless: private communication.

of a fluon diaphragm backed by rubber, placed at the end of a $\frac{3}{8}$ in. dia. $\times 8$ in. pipe along which a temperature gradient is maintained. The chamber temperature is controlled by closely wrapped electrical heating tape, and measured by thermocouples placed in the metal body $\frac{1}{4}$ in. from the top and bottom of the liquid.

3. – Methyl iodide-propane system.

Various percentages by volume of methyl iodide in propane were introduced into the chamber by distillation and the operating range of sensitivity determined. A 10 millicurie ^{60}Co source was used for most of the work but in a few cases gammas from a 29 MeV microtron provided the ionizing particles. The results are shown in Fig. 1 as a plot of operating temperature against equilibrium pressure, the solid lines representing the limits of sensitivity for the system.

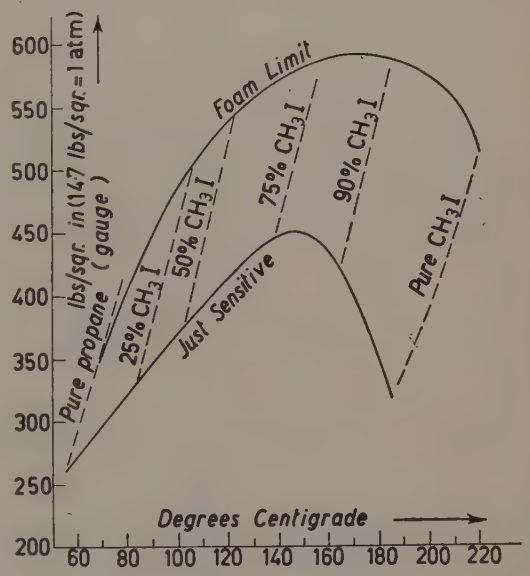

Fig. 1.

Propane was found to be a successful inhibitor only in large concentrations. For instance a 50%-50% mixture by volume remained completely colourless for only a short period once the operating temperature (115 °C) was reached. Continuous operation was possible only for about an hour by which time the liquid was pink enough seriously to limit the photographic reproduction. However, after subsequently leaving the liquid to cool to room temperature it was found to clear, the free iodine being deposited as a film around the chamber walls. Higher CH_3I concentrations could be operated for correspondingly shorter times, whilst the pure liquid was already a deep red colour by the time operating temperature had been attained.

4. – Tri-fluoro bromo methane (CF_3Br).

Because of their low critical temperatures and pressures certain freons offer attractive possibilities as bubble chamber liquids [4]. Further, it is possible to obtain brominated freons, which, whilst retaining the convenient oper-

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[4] G. G. Blinov, Iu. S. Krestnikov, M. A. Lomanov and Ia. Ia. Shalamov: *Žu. Eksp. Teor. Phys.*, **5**, 1281 (1957).

ating conditions also have fairly short radiation lengths [5]. We have investigated in particular tri-fluoro bromo methane (trade name Isceon B.1) and have determined its critical temperature and pressure to be (67 ± 1) °C and (595 ± 8) lbs/sq. in. respectively. Sensitivity of this liquid to ionizing radiations was found to commence at 22 °C whilst the foam limit occured at 34 °C.

5. – Conclusion.

The radiation length of a 50%-50% by volume mixture of CH_3I and C_3H_8 at its operating temperature is about 10 cm. However, a 25% CH_3I concentration, which is almost the highest concentration that can be operated for any reasonable length of time without obvious discolouration, has a radiation length of about 17 cm. This compares with a value of 11.0 cm for CF_3Br at 30 °C. The latter substance is completely stable at its operating temperature and prolonged exposure to light has produced no observable discolouration. In addition, it is non-toxic, non-inflammable, and non-corrosive. As a bubble chamber liquid it is therefore ideal, and would seem to be much preferable to the CH_3I—C_3H_8 system which, besides its less convenient operating conditions, also presents difficulties in mixing and in retaining uniform concentration, disadvantages inherent in gas-liquid systems.

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[5] D. V. Bugg: private communication.

RIASSUNTO (*)

Si è esaminata la convenienza di un sistema ioduro di metile-propano come riempitivo di una camera a bolle a « liquido pesante ». Si sono trovate le condizioni di funzionamento per diverse miscele ed anche per il trifluoro bromo metano, altro possibile liquido pesante di riempimento.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. X, N. 4 16 Novembre 1958

The Multipactor Effect in a Linear Electron Accelerator.

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(ricevuto l'11 Agosto 1958)

Summary. — Phenomena, due to the multipactor effect and observed in the linear electron accelerator of the Ghent University are reported. An explanation is proposed for the variation of the anomalous attenuation due to the multipactor effect as a function of the R.F. power fed in the accelerator.

While operating the linear electron accelerator at the Ghent University the following interesting phenomena, were observed.

1) As soon as the magnetron is switched on and R.F. power is fed into the machine, the gas pressure in the apparatus increases by a factor of about three. Then it decreases gradually and returns to its original value after several tens of minutes. Often discharges occurred in some places in the accelerator and waveguide structures, especially at the input « door-knob » coupling. If the R.F. power is turned off for a few minutes and then turned on again, the highest value of the pressure is less than before and returns faster to its original value. Finally after about ten cycles of turning the R.F. power on and off, the pressure is no longer influenced by the R.F. power.

2) If the focusing magnetic field H_f is switched on while the magnetron is running, the pressure rises again. A variation of the current in the focusing coils changes the gas pressure in the accelerator tube. The time needed to return to a pressure slightly higher than the original pressure, decreases for every next run, but never becomes zero.

3) H_f causes an anomalous attenuation of the R.F. power if this is at the normally used level of 0.6 mW. If very low power levels are used, no such effects are observed. For power levels of $\simeq$ 100 till 600 kW and with a constant focusing field, this anomalous attenuation is not a monotone function of the input power.

4) At the normal R.F. power level and with H_f on, a rather high amount of X-rays is observed over the whole lenght of the accelerator tube.

5) After the accelerator has been in operation for many hours, the discs of the cavities show several concentric rings, differently coloured. The diameters of these rings are different from cavity to cavity, but on all discs, the rings cease to appear at a few millimeter from the edge near the outside wall of the tube. The rings are most pronounced near the R.F. power input. It must be noted that all the cavities have different dimensions and are made to have a $\pi/2$ mode travelling wave with a constant effective longitudinal electric field of 20 kV/cm.

6) The accelerator operates best at a frequency slightly different from the frequency for which it was designed and matched.

All these phenomena were reproducible as far as the effects themselves were concerned. Doing some rough measurements, however, lead nearly each time to different numerical results.

Phenomena, similar to these described here, were also observed in some accelerators constructed in England ([1-5]). It was pointed out by MULLETT, CLAY and HADDEN ([3]) and by HADDEN ([4]) that the mentioned troubles were worse for dielectric loaded linear accelerators than for all-metal machines, and that they were less serious if mercury diffusion pumps were used instead of oil pumps. In the latter case, the kind of oil influenced the effect. In this regard it should be mentioned that the linear accelerator in Ghent has an all-metal corrugated wave-guide, that we had silicon oil in the diffusion pumps and that no cold traps were used.

MULLETT *et al.* ([3]) explain the phenomena observed in accelerators on the basis of the multipactor effect. It is an effect associated with surfaces whose secondary emission coefficient η is greater than one, under the conditions that the effect occurs. The multipactor effect was first described by FARNSWORTH ([6,7]), was further developed by several authors ([8-13]) and especially by KREBS and MEERBACH ([14-16]). None of these papers, however, contains anything with relation to linear accelerators. They all consider mainly the case

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([1]) R. B. R. SHERSBY-HARVIE and L. B. MULLETT: *Proc. Phys. Soc.*, B **62**, 270 (1949).
([2]) C. W. MILLER and G. SAXON: *Nature*, **172**, 1163 (1953).
([3]) L. B. MULLETT, R. E. CLAY and R. J. B. HADDEN: AERE GR/R 1076 (1953).
([4]) R. J. B. HADDEN: AERE G/R 1161 (1953).
([5]) G. B. WALKER and E. L. LEWIS: *Nature*, **181**, 38 (1958).
([6]) P. T. FARNSWORTH: *Journ. Franklin Inst.*, **2**, 411 (1934).
( ) P. T. FARNSWORTH: *Zeits. Phys.*, **129**, 491 (1951).
([8]) E. W. B. GILL and A. VON ENGEL: *Proc. Roy. Soc.*, A **197**, 107 (1949).
( ) E. W. B. GILL and A. VON ENGEL: *Proc. Roy. Soc.*, A **192**, 446 (1948).
([10]) A. J. HATCH and H. S. WILLIAMS: *Journ. Appl. Phys.*, **25**, 417 (1954).
([11]) B. LAX, W. P. ALLIS and S. C. BROWN: *Journ. Appl. Phys.*, **21**, 1297 (1950).
([12]) W. G. ABRAHAM: *Stanford Microwave Laboratory Thesis* (1950).
([13]) F. KOSSEL and K. KREBS: *Zeits. Phys.*, **139**, 189 (1954).
([14]) K. KREBS: *Zeits. Angew. Phys.*, **2**, 400 (1950).
([15]) K. KREBS and F. MEERBACH: *Ann. Phys.*, **15**, 189 (1955).
([16]) K. KREBS and F. MEERBACH: *Ann. Phys.*, **18**, 146 (1956).

of two parallel flat plates with a R.F. electric field, normal to the plates. A simple representation can be given as follows. An electron comes out of one of the plates (*e.g.* by field-emission) at a time t and is accelerated towards the other plate so that it arrives there at a time $t+(2n+1)/2f$, f being the frequency of the R.F. field and n being an integer. The secondary electrons find the same situation at $t+(2n+1)/2f$, as the first electron at time t, (if there is no time-delay between the arrival of an electron and the emission of the secondaries). If $\eta<1$, there will be a rapid build-up.

The phenomenon of the coloured rings must be due to thin oil-films on the iris-walls. The multipactor effect itself produces cracking products, which can also give rise to a large increase of η. The mechanism of the phenomenon has been studied and described by AITKIN ([17]).

Since no higher power source was available, we have not been able to feed more than 0.6 mW in the linear accelerator. We should have liked to do so, in view of an experiment reported by MILLER and SAXON ([2]). They measured in a linear accelerator an attenuation of 2.5 db for all power levels, in the absence of the magnetic focusing field. In the presence of such fields, the attenuation was still found to be of the order of 2.5 db for power fluxes of a few mW and several MW. There was however a variation of attenuation with power in an intermediate range. They noted a maximum of 12 db for an input power of 250 kW. If the power level is increased, the attenuation falls, approaching 2.5 db for an input of 3 MW. Since, in our mind, no satisfactory explanation has been given for it, we want to turn our attenuation to this phenomenon of the dependence of attenuation on power level.

Let us consider the condenser type electric field $E\sin(\omega t+\varphi)$, in which $\omega=2\pi f$, $\varphi=$ phase angle at which the electron is emitted. Furthermore m and e are the mass and charge of the electron and d is the distance between the plates. In order to have multipactoring, the condition.

$$d=\frac{e}{m}\cdot\frac{E}{\omega^2}[(2n+1)\pi\cos\varphi+2\sin\varphi],$$

must be satisfied. The requirement of phase stability or bunching, (then the multipactoring can occur in a stable way), leads to

$$0<\operatorname{tg}\varphi<2/\pi(n+1),$$

so that for each n, E has to lie between two extreme values: $E_{\min}$ and $E_{\max}$. $E_{\min}$ corresponds to $\varphi_{\min}=\operatorname{arctg}[2/\pi(2n+1)]$ and $E_{\max}$ corresponds to $\varphi=0$. Table I gives these values for the case $\lambda=c/f=10.00$ cm, $d=2.50$ cm and $n=0.1\ldots,5$. It gives also $\varphi_{\min}$ and the extreme values of the energy ε with which the electrons can arrive at the opposite plate. We have

$$\varepsilon=2d^2\omega^2m\frac{\cos^2\varphi}{[(2n+1)\pi\cos\varphi+2\sin\varphi]^2}.$$

([17]) D. AITKIN: To be published in *Proc. Inst. Electr. Eng.*, **105** B (1958).

TABLE I.

n	φ_{min}	E_{min} (kV/cm)	E_{max} (kV/cm)	ε_{min} (keV)	ε_{max} (keV)
0	32° 19′	134.3	159.2	128.1	253.1
1	11° 59′	51.90	53.05	25.65	28.13
2	7° 15′	31.575	31.835	9.81	10.13
3	5° 12′	22.643	22.735	5.094	5.162
4	4° 03′	17.640	17.685	3.094	3.125
5	3° 19′	14.445	14.468	2.075	2.094

The regions in which the multipactoring is possible in a stable way, are rather narrow. For increasing n, these regions become smaller and smaller and the fields needed decrease. ε decreases also and becomes so small that the multipactoring will be impossible because of the fact that the secondary emission coefficient becomes less than one, for low enough values of ε. It is clear that in reality the limit values given in Table I are not very sharp. Among other things, this is due to the fact that the initial velocities v_0 of the electrons have a statistical distribution. (In the derivation of the above formulae, v_0 was supposed to be zero).

The electric field E_z parallel with the axis of the accelerator is, for $a \leqslant r \leqslant b$, given by

$$E_z(r) = E_a \frac{J_0(kr)Y(kb) - Y_0(kr)J_0(kb)}{J_0(ka)Y(kb) - Y_0(ka)J_0(kb)},$$

in which J_0 and Y_0 are Bessel functions, $k = 2\pi/\lambda$ and $E_a = (E_z)_{r=a}$ The meaning of the other symbols is clear from Fig. 1. This function E_z has been drawn on Fig. 2 for $\lambda = 10.00$ cm, $a = 1.30$ cm and $b = 4.00$ cm. E_z is approximately a decreasing linear function of r. For a given value of n the multipactor effect can occur only in a ring whose radii r_1 and r_2 are defined by $E_1 = E_{max}$ and $E_2 = E_{min}$. It is easily seen that such a ring is very narrow, especially for $n \neq 0$.

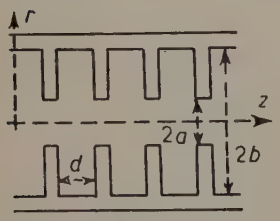

Fig. 1.

The fact that the anomalous attenuation is measured only in the presence of the focusing field can be explained satisfactorily. In the corrugations, there is not only an electric field E_z, but there is also a component H_θ, causing a radial force on the electrons, which pushes the electrons away from the regions where the multipactoring is possible in a stable way. The axial magnetic focusing field tends to prevent electrons from moving out of these regions.

If the power flux through the accelerator is increased, the values of E_1 and E_2 are unchanged. The interval of $r(r_1, r_2)$ where the effect can occur, moves however toward the outside and falls off. This is illustrated by Fig. 2.

Let us suppose that the curves of this figure are straight lines, given by the equation

$$E = \beta(b - r), \qquad (\beta > 0);$$

Then, the area S of the ring where the multipactor effect is possible, is

$$(1) \qquad S = \begin{cases} 0 & (b-a)\beta \leqslant E_2, \\ \pi\left[\left(\frac{E_2}{\beta}-b\right)^2 - a^2\right] & E_2 \leqslant \beta(b-a) \leqslant E_1, \\ \pi\left[\left(\frac{E_2}{\beta}-b\right)^2 - \left(\frac{E_1}{\beta}-b\right)^2\right] & E_1 \leqslant \beta(b-a). \end{cases}$$

It is obvious that, for any particular n-value, S is directly proportional to the energy loss. Fig. 3 gives $S(\beta)$ for $a = 1.30$ cm, $b = 4.00$ cm, $\lambda = 10$ cm, $d = 2.5$ cm and $n = 1$. S increases fast with β, in the interval $[E_2/(b-a), E_1/(b-a)]$. This interval is very small. For $\beta > E_1/(b-a)$, S is first still increasing but very much slower, reaches a maximum for $\beta = (E_1+E_2)/b$, and then falls off, going to 0 for $\beta \to \infty$.

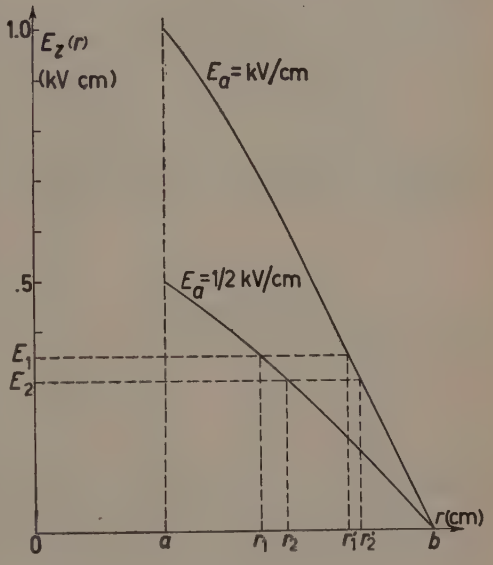

Fig. 2.

For very low β, only high values of n could give rise to multipactoring, no high fields being required in that case. The effect will not occur, however, because of the fact that ε is too low to have $n > 1$. If β is increased, lower n values will enter the picture, for which also ε is big enough to have $\eta > 1$. Increasing the power flux still more, will bring in more rings, the area of which is each time larger than the area of the one before. The intervals between the power needed for having successive rings increase very rapidly. Even if the power were raised so much that the very low n-values could be reached, no more multipactor rings would come in because ε would be so high that η would be less than one.

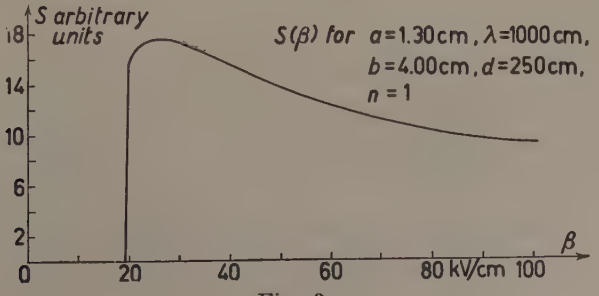

Fig. 3.

The most part of the energy loss, caused by multipactoring, must be due to the inner ring, corresponding to the lowest n-value reached in any particular accelerator and which can give rise to the effect. This is because of the larger area involved and also because of the larger value of ε.

Let us now consider the results given by MILLER and SAXON. For 250 kW input 10 db of attenuation is due to the multipactor effect. By increasing the power till 3 MW, the electric field will become $\sqrt{12}$ times larger. From Table I one can see that no such large intervals exist between two successive multipactoring E-regions. However, the next ring corresponding to a lower n-value, possible as far as the needed electric field is concerned, will not give rise to multipactoring because of the high ε value ($\eta < 1$). Taking into account only the ring corresponding to the lowest n, that gives rise to multipactoring, and supposing that there is only one cavity, in which all losses occur, this attenuation occurs for $\beta = (E_1 + E_2)/b$. One finds then $P_0 = 10P_u$ ($P_0 =$ input power, $P_u =$ output power), and $S = \pi b^2(E_1 - E_2)/(E_1 + E_2)$. For 3 MW one has $P' = 12P_0$, $\beta' = \sqrt{12}\,\beta$ and $S' = S(2\sqrt{12} - 1)/12$. From this one finds easily that the attenuation is now only 0.157 db.

One comes somewhat nearer to reality if an accelerator with a purely periodic structure, without other losses than those due to multipactoring, is considered. Because of these losses each cavity has a different value for β. One can write $\beta = \beta(z)$. Again we consider only one n-value. Using

$$\mathrm{d}P \sim \beta\,\mathrm{d}\beta = -C'S\,\mathrm{d}z\,, \qquad (C' = \text{constant}),$$

and the third expression for S, one finds

$$\mathrm{d}\beta = C\left(\frac{E_1 - E_2}{\beta^3} - \frac{2b}{\beta^2}\right)\mathrm{d}z\,, \qquad (C = \text{constant}),$$

so that the dependance of β on z along the accelerator is given by

$$(2) \qquad z = \frac{1}{2bC}\left\{\frac{\beta^3(0) - \beta^3}{3} + \frac{E_1 + E_2}{4b}[\beta^2(0) - \beta^2] + \right.$$
$$\left. + \frac{(E_1 + E_2)^2}{4b^2}[\beta(0) - \beta] + \frac{(E_1 + E_2)^3}{8b^3}\lg\frac{\beta(0) - (E_1 + E_2)/2b}{\beta - (E_1 + E_2)/2b}\right\}.$$

Maximum attenuation will occur approximately for

$$\beta(l) = 3(E_1 + E_2)/4b\,.$$

($l =$ lenght of the accelerator). Taking into account the value of a/b for most accelerators, this is roughly the value for which the multipactor ring is just in the last cavity. $P_0 = 250$ kW in the considered case and for an attenuation of 10 db one has $\beta(0) = 3\sqrt{10}(E_1 + E_2)/4b$. Using (2), this leads to

$$(3) \qquad 2bCl\left(\frac{b}{E_1 + E_2}\right)^3 = 6.229\,.$$

If now $P_0' = 3$ MW then $\beta'(0) = \sqrt{12}\,\beta(0)$. Using (2) and (3) one finds an equation for $\beta(l)$ leading to

$$\beta(l) = 8.126\,\frac{E_1 + E_2}{b}\,, \qquad \text{and } \log\frac{P_0}{P_u} = 2\log\frac{8.215}{8.126} = 0.022\,.$$

Thus we have 0.22 db attenuation for 3 MW input. This agrees well with the experiment.

We want to point out that the decrease in attenuation is mostly all due to the increase of P_0. Even if the absolute power losses were constant for different power levels, we would find a low attenuation (0.34 db) for 3 MW if one supposed to have 10 db for 250 kW input.

Finally it is interesting to remark that the increase of the input power will also increase the radial forces which try to push the electrons away from the stable region, while E remains constant in the considered ring. This will decrease the losses due to the multipactor effect even more.

* * *

We wish to thank the Interuniversity Institute for Nuclear Sciences (Belgium) that sponsored this work.

RIASSUNTO (*)

Si riferiscono fenomeni dovuti all'effetto multipactor e osservati nell'acceleratore lineare dell'università di Gent. Si propone una spiegazione per la variazione dell'attenuazione anomala dovuta all'effetto multipactor in funzione dell'energia a r.f. con la quale si alimenta l'acceleratore.

(*) *Traduzione a cura della Redazione.*

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

Self-Energy of the Longitudinal Neutrino.

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(ricevuto il 18 Settembre 1958)

As is well-known the longitudinal neutrino theory is based on the assumption that the physical mass of the neutrino and the anti-neutrino with the inclusion of all interactions is strictly zero [1]. The question then arises, in comparison to the photon self-energy, if it is possible to demonstrate that the self-energy of the longitudinal neutrino (and the anti-neutrino) vanishes by virtue of the so-called neutrino gauge. We wish to show by direct calculation that the self-energy in question vanishes in the lowest order perturbation calculation, in which the interactions involving neutrinos are taken to be those containing four Fermions represented by the Puppi triangle [2].

First we take the β-decay interactions. The interaction Lagrangian density may be written [1]

$$L(x) = \sum_{i=1}^{5} \big[C_i \, (\bar{\psi}_{\mathrm{p}}(x) O_i \, \psi_{\mathrm{n}}(x)) (\bar{\psi}_{\mathrm{e}}(x) O_i \, (1 - \gamma_5) \psi_{\nu}(x)) + \\ + C_i^* \, (\bar{\psi}_{\mathrm{n}}(x) O_i \, \psi_{\mathrm{p}}(x)) (\bar{\psi}_{\nu}(x) (1 + \gamma_5) O_i \, \psi_{\mathrm{e}}(x)) \big] ,$$

where $O_1 = 1$, $O_2 = \gamma_\alpha$, $O_3 = \sigma_{\alpha\beta}$, $O_4 = \gamma_\alpha \gamma_5$, and $O_5 = \gamma_5$.

The S-matrix corresponding to the lowest order self-energy diagram ($\nu \rightleftarrows \mathrm{p} + \bar{\mathrm{n}} + \mathrm{e}^-$) is then found to be

$$S = (2\pi)^{-4} \delta(p' - p) \sum_i \sum_j C_i \, C_j^* \int (\mathrm{d}^4 k_1)(\mathrm{d}^4 k_2) \cdot \\ \cdot (\bar{U}_{\nu}(1 + \gamma_5) O_j \, (\gamma p - \gamma k_1 + \gamma k_2 + i m_{\mathrm{e}}) O_i \, (1 - \gamma_5) U_{\nu}) \cdot \\ \cdot \mathrm{trace} \big[(\gamma k_1 + i m_{\mathrm{p}}) O_i (\gamma k_2 + i m_{\mathrm{n}}) O_j \big] \cdot \big[\{ (p - k_1 + k_2)^2 + m_{\mathrm{e}}^2 \} (k_1^2 + m_{\mathrm{p}}^2)(k_2^2 + m_{\mathrm{n}}^2) \big]^{-1} ,$$

where m_{e}, m_{p}, and m_{n} are the masses of the electron, proton, and neutron. p and p' are the energy-momentum 4-vectors of the neutrino before and after the self-energy interaction, and U_{ν} is the Dirac spinor for the neutrino.

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[1] L. LANDAU: *Nucl. Phys.*, **3**, 127 (1957); A. SALAM: *Nuovo Cimento*, **5**, 299 (1957): T. D. LEE and C. N. YANG: *Phys. Rev.*, **105**, 1671 (1957).

[2] G. PUPPI: *Nuovo Cimento*, **5**, 505 (1948); also for the general discussion and other references, M. GELL-MANN and A. H. ROSENFELD: *Ann. Rev. Nucl. Sci.*, **7**, 407 (1957).

For the purpose of analysing the above expression we will first transform the integrand in following steps. First making use of the identities

$$(ab)^{-1} = \int_0^1 \mathrm{d}z[az + b(1-z)]^{-2},$$

$$(ab^2)^{-1} = 2\int_0^1 \mathrm{d}x\,(1-x)\,[ax + b\,(1-x)]^{-3},$$

we may write $(q = k_1 - k_2)$

$$\big[((p-q)^2 + m_e^2)(k_1^2 + m_p^2)(k_2^2 + m_n^2)\big]^{-1} =$$

$$= 2\int_0^1 z(1-z)\,\mathrm{d}z \int_0^1 (1-x)\,\mathrm{d}x\big[z(1-z)(q-px)^2 + xz(1-z)m_e^2 +$$

$$+ \{(k_2 + qz)^2 + m_n^2(1-z) + m_p^2 z\}(1-x)\big]^{-3}.$$

Then introducing variables a and b defined by $k_2 + qz = a$ and $q - px = b$, we obtain

$$S = (2\pi)^{-4}\delta(p'-p)\sum_i\sum_j C_i C_j^* \int_0^1 z(1-z)\,\mathrm{d}z \int_0^1 (1-x)\,\mathrm{d}x \int (\mathrm{d}^4a)(\mathrm{d}^4b)\cdot$$

$$\cdot(\bar{U}_\nu(1+\gamma_5)O_j\{(1-x)\gamma p + im_e - \gamma b\}O_i(1-\gamma_5)U_\nu)\cdot$$

$$\cdot\mathrm{trace}\big[(\gamma a)\,O_i\,(\gamma a)O_j + \{\gamma b + (pb)(1-z)x + im_p\}O_i\{-z(\gamma b) - zx(\gamma p) + im_n\}O_j\big]\cdot$$

$$\cdot\big[z(1-z)\,b^2 + xz(1-z)m_e^2 + (1-x)(a^2 + m_n^2(1-z) + zm_p^2)\big]^{-3}.$$

Let us first take $i=j=1$, or $O_i = O_j = 1$. Then, one may set

$$(\bar{U}_\nu(1+\gamma_5)\{(1-x)\gamma p + im_e\}(1-\gamma_5)U_\nu) \equiv 0,$$

immediately, since $(\gamma p)\,U_\nu \equiv 0$ because of the zero rest mass of the neutrino. If the neutrino were not longitudinal, then its wave function would not be in the form of $(1-\gamma_5)\,\bar{U}_\nu$, and therefore the above expression would not vanish even for the vanishing neutrino rest mass. For the longitudinal neutrino we then find that the integrand is proportional to $\big(\bar{U}_\nu(1+\gamma_5)(\gamma b)(pb)(1-\gamma_5)U_\nu\big)$, which, when averaged over directions of the 4-vector b, becomes $\frac{1}{4}\big(\bar{U}_\nu(1+\gamma_5)b^2(\gamma p)(1-\gamma_5)U_\nu\big)$. Thus, we find that the above particular term $(i=j=1)$ is of the form of the product of the rest mass of the neutrino and a certain divergent integral, which may be regarded as zero. It can be easily shown that the remaining terms, *i.e.* $i=j\neq 1$ and the cross terms, each vanish in similar manner. This situation persists with regard to other interactions in the Puppi triangle, namely $\nu \rightleftarrows p + \bar{n} + \mu^-$ and $\nu \rightleftarrows \mu^- + e^+ + \nu$. Thus, the self-energy of the longitudinal neutrino arising from the interactions involving four fermions vanishes in the lowest order perturbation calculation by virtue of the neutrino gauge.

IL NUOVO CIMENTO Vol. X, N. 4 16 Novembre 1958

Some Remarks on the Quantum Mechanics of a Lattice Gas.

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(ricevuto il 27 Settembre 1958)

It is shown that there exist two different Hamiltonians for a lattice gas which correspond to the same Hamiltonian of a system of N particles in the theory of second quantization. Calculations based on these two Hamiltonians lead obviously to different results. It seems a priori impossible to find theoretical arguments in favour of one of them.

Following Matsubara, Matsuda [1] and Morita [2] we consider the lattice gas in which N molecules are on L lattice sites $(r_1 \dots r_L)$. We use here arbitrary lattices satisfying the following conditions:

a) there exist vectors η_i, $i = (-z/2, \dots, -1, 1, \dots, z/2)$ where z is the co-ordination number of the lattice; such that:

$$\eta_i + r_k = r_{k+n_{ki}} \qquad \text{and} \qquad |\eta_i| = d = \min_{s \neq m} |r_s - r_m| .$$

b) $\eta_i = -\eta_{-i}$; $\eta_i \neq \eta_j$ if $i \neq j$. We define an operator of the ratio of differences:

$$\Delta_{k,i} = \frac{1}{d}(T_{k,i} - 1); \qquad T_{k,i}\Psi(\dots r_k \dots) = \Psi(\dots r_k + \eta_i \dots) . \tag{1}$$

We have (because of *b*))

$$T_{k,i}\, T_{k,-i} = 1 . \tag{2}$$

From (1) we have:

$$\Delta_{k,i}^2 = \frac{1}{d^2}(T_{k,i}^2 + 1 - 2T_{k,i}) . \tag{3}$$

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[1] T. Matsubara and H. Matsuda: *Progr. Theor. Phys.*, **16**, 596 (1956) and **17**, 19 (1957).
[2] T. Morita: *Progr. Theor. Phys.*, **18**, 462 (1957).

We take into consideration only state functions ψ defined on the set of lattice points r_k. It may be easily seen that by taking $\eta_i = d(\delta_{i1}, \delta_{i2}, \delta_{i3})$ for $i = 1, 2, 3$ and $\eta_{-i} = -\eta_i$ we get vectors possessing the properties *a*) and *b*) for a simple cubic lattice. By virtue of (3):

$$\Delta^2_{k,i}\,\psi(\ldots x_k \ldots) = \frac{1}{d^2}\left[\psi(\ldots x_k + 2d \ldots) - 2\psi(\ldots x_k + d \ldots) + \psi(\ldots x_k \ldots)\right].$$

If we compare the Morita definition of the operator $\Delta^2/\Delta x_k^2$ with this result, we see that:

$$\Delta^2_{k,1} = T_{k,1}\frac{\Delta^2}{\Delta x_k^2}.$$

Taking into account the correspondence with the continuous case we see that the kinetic energy ought to be proportional to the sum of squares of displacement operators changing the argument of the state function by η_i. Thus for the operator $\Delta/\Delta x_k$ such that $(\Delta/\Delta x_k)^2 = \Delta^2/\Delta x_k^2$ we have:

$$\frac{\Delta}{\Delta x_k} = T^{\frac{1}{2}}_{k,-1}\Delta_{k,1}.$$

where

$$T^{\frac{1}{2}}_{k,-1}\psi(\ldots x_k \ldots) = \psi(\ldots x_k - d/2 \ldots).$$

As we see the operator $\Delta/\Delta x_k$ operating on the state functions creates functions which are not state functions since they depend on points which are not lattice points. Thus it is seen that the Morita operator of kinetic energy cannot be treated as the sum of squares of operators in this space.

We shall now show that to the two equivalent hamiltonians

$$\frac{\hbar^2}{2m}\int \nabla\psi^* \nabla\psi(\mathrm{d}r) \quad \text{and} \quad -\frac{\hbar^2}{2m}\int \psi^* \nabla^2\psi(\mathrm{d}r),$$

correspond two essentially different hamiltonians for the lattice gas.

We replace here the expression $\nabla\psi^* \nabla\psi$ by $\sum_{i=1}^{z/2} \Delta_{k,i}\psi^*\Delta_{k,i}\psi$, the expression $\psi^* \nabla^2\psi$ by $\sum_{i=1}^{z/2} \psi^*\Delta^2_{k,i}\psi$, and the integral by the sum over the lattice points. We obtain:

$$\frac{\hbar^2}{2m}\int \nabla\psi^* \nabla\psi(\mathrm{d}r) \to \frac{\hbar^2}{2m}\sum_{k=1}^{L}\sum_{i=1}^{z/2} \Delta_{k,i}\psi^*(r_k)\Delta_{k,i}\psi(r_k), \tag{4}$$

$$-\frac{\hbar^2}{2m}\int \psi^* \nabla^2\psi(\mathrm{d}r) \to -\frac{\hbar^2}{2m}\sum_{k=1}^{L}\sum_{i=1}^{z/2} \psi^*(r_k)\Delta^2_{k,i}\psi(r_k). \tag{5}$$

The expressions (4) and (5) go over into hamiltonians for the continuous position spectrum when d tends to zero, only for simple cubic lattices. This is caused by

a linear dependence of vectors η_i, $i=1, \ldots, z/2$. On the other hand in expressions (4) and (5) we must take into account summations over all displacements η_i, $i=1, \ldots, z/2$.

Let us consider the expression (4). We take after Morita:

$$\psi(r_k) = \sum_{j=1}^{L} \delta_{r_j, r_k} a_j ,$$

where the operators a_j and a_j^* fulfill the commutations relations given by Matsubara and Matsuda. We obtain:

$$(4') \quad \frac{\hbar^2}{2md^2} \sum_{k=1}^{L} \sum_{i=1}^{z/2} \Big[\sum_{j=1}^{L} (\delta_{r_j, r_k+\eta_i} a_j^* - \delta_{r_j, r_k} a_j^*)\Big] \Big[\sum_{m=1}^{L} (\delta_{r_m, r_k+\eta_i} a_m - \delta_{r_m, r_k} a_m)\Big] =$$

$$= \frac{\hbar^2}{2md^2} \sum_{k=1}^{L} \sum_{i=1}^{z/2} (a_{k+n_{ki}}^* - a_k^*)(a_{k+n_{ki}} - a_k) = \frac{\hbar^2}{2md^2} \sum_{\langle i, j \rangle} (a_i^* - a_j^*)(a_i - a_j) .$$

Summation over $\langle i, j \rangle$ denotes summation over all the nearest pairs of points. For simple cubic lattices the hamiltonian (4′) equals that given by Matsubara and Matsuda.

Let us consider the expression (5). We obtain:

$$(5') \quad -\frac{\hbar^2}{2md^2} \sum_{k=1}^{L} \sum_{i=1}^{z/2} \Big[\sum_{j=1}^{L} a_j^* \delta_{r_j, r_k}\Big] \Big[\sum_{m=1}^{L} (a_m \delta_{r_m, r_k+2\eta_i} - 2a_m \delta_{r_m, r_k+\eta_i} + a_m \delta_{r_m, r_k})\Big] =$$

$$= -\frac{\hbar^2 z}{4md^2} \sum_{k=1}^{L} a_k^* a_k + \frac{\hbar^2}{md^2} \sum_{\langle i, j \rangle} a_i^* a_j - \frac{\hbar^2}{2md^2} \sum_{k=1}^{L} \sum_{i=1}^{z/2} a_k^* a_{k+l_{ki}} ,$$

where l_{ki} denotes the index of the vector $r_k + 2\eta_i$. As we see the expression obtained in this way is quite different from (4′). It is interesting that the expression contains interactions with lattice points with the distance $2d$ as well, and it does not contain the interaction terms describing interactions with lattice points at distances smaller than $2d$ and larger than d.

There appears the question why these two lattice point hamiltonians are not equivalent. This is connected with the fact, that in difference calculus:

$$\Delta(ab) = (\Delta a)b + (Ta)(\Delta b) .$$

Hence:

$$(6) \quad O = \sum_{i=1}^{z/2} \sum_{k=1}^{L} \Delta_{k,i} \left(\psi^*(r_k) \Delta_{k,i} \psi(r_k)\right) =$$

$$= \sum_{i=1}^{z/2} \sum_{k=1}^{L} \Delta_{k,i} \psi^*(r_k) \Delta_{k,i} \psi(r_k) + \sum_{i=1}^{z/2} \sum_{k=1}^{L} \left(T_{k,i} \psi^*(r_k)\right) \Delta_{k,i}^2 \psi(r_k) .$$

Hence:

$$(7) \quad \frac{\hbar^2}{2m} \sum_{i=1}^{z/2} \sum_{k=1}^{L} \Delta_{k,i} \psi^*(r_k) \Delta_{k,i} \psi(r_k) = -\frac{\hbar^2}{2m} \sum_{i=1}^{z/2} \sum_{k=1}^{L} T_{k,i} \left(\psi^*(r_k) T_{k,-i} \Delta_{k,i}^2 \psi(r_k)\right) .$$

The last equality is the consequence of (6), (2), and of the connexion $T(ab) = TaTb$. The expression (7) up to the operators a_j and a_j^* at the band-points of the lattice

is identical with:

$$-\frac{\hbar^2}{2m}\sum_{i=1}^{z/2}\sum_{k=1}^{L}\left(\psi^*(r_k)T_{k,-i}\,\Delta_{k,i}^2\,\psi(r_k)\right).$$

This expression for the simple cubic lattice equals the hamiltonian given by Morita.

By elementary computations we obtain the result that hamiltonians (4′) and (5′) commute with the total number of particles. Thus we have no reasons for preferring one of them.

It may be noted that the transition from the quantum mechanics with continuous position spectrum to quantum mechanics of lattice gas is very difficult, because of two reasons: (i) the classical analogue does not exist and (ii) it is impossible to use the canonical formalism, because of:

$$[\Delta_{k,1}, x_i] = \delta_{ik}\,T_{k,1}.$$

On the other hand, as shown in this note, a procedure based on the correspondence with the case of continuous position spectrum is not unique.

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Pion-Proton Scattering Dispersion Relations.

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(ricevuto il 20 Ottobre 1958)

We have re-examined the Puppi-Stanghellini (P.S.) discrepancy in the π-p scattering dispersion relations [1].

The main difference between our analysis and that of P.S. is the use of different values of the low energy π^--p total cross sections for the evaluation of the integrals. The data which P.S. used for drawing the low energy total cross section were: (12.9 ± 1.7) mb at 37 MeV [2], (17.6 ± 2.2) mb at 58 MeV [3] and the two values (21 ± 8) mb at 110 MeV and (31 ± 9) mb at 113 MeV [4], the last two with large errors: P.S. did not use the cross section (14.7 ± 1.8) mb at 65 MeV [5], obtained by integration, which was in disagreement with the other data.

Since the P.S. analysis, other experiments have been finished. These new total cross section data yield (7.8 ± 0.7) mb at 41.5 MeV [6] and (21.1 ± 0.6) mb at 98 MeV [7]. Taking these new data into account, the best fit to the ensemble of total cross sections below the resonance is substantially lower than the cross sections used by P.S. It follows that also the 65 MeV point is in good agreement with the other data, and the only point remaining in disagreement is the 58 MeV value. This change has a negligible effect on the D^+ dispersion relation, owing to the fact that the integral for D^+ involves only σ^- divided by $\omega' + \omega$. But the modification may be noticeable in the D^- dispersion relation, especially in the region in which the slope of σ^- is increased.

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[1] G. Puppi and A. Stanghellini: *Nuovo Cimento*, **5**, 1305 (1957).

[2] C. E. Angell and J. P. Perry: *Phys. Rev.*, **92**, 835 (1953).

[3] P. J. Isaacs, A. M. Sachs and J. Steinberger: *Phys. Rev.*, **85**, 803 (1952).

[4] H. L. Anderson, E. Fermi, E. A. Long, R. Martin and P. E. Nagle: *Phys. Rev.*, **85**, 934 (1952).

[5] D. Bodansky, A. M. Sachs and J. Steinberger: *Phys. Rev.*, **93**, 1367 (1954).

[6] S. Barnes, B. Rose, G. Giacomelli, J. Ring and K. Miyake: University of Rochester Report NYO-2170.

[7] J. R. Holt: private communication to prof. Puppi.

The result of the calculation of D^- with these new data is shown in Fig. 1, in which the two points obtained from the angular distributions at 41.5 MeV and 98 MeV have been added. In Fig. 1 the curve calculated independently by SALZMAN [8] using the Anderson-Piccioni cross-sections [9,10] is also shown. The good agreement between the two curves is due to the fact that the new total cross sections used by us are consistent with the Anderson phase shifts.

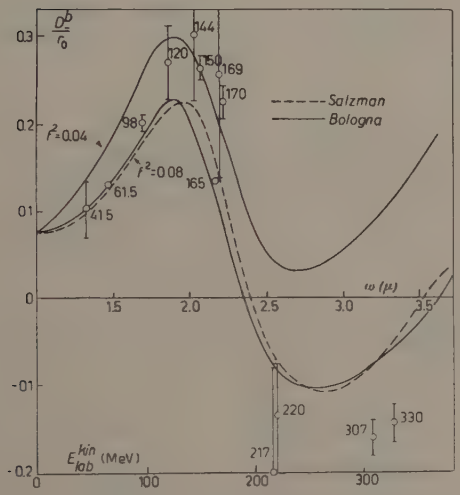

Fig. 1. – D^b_-/r_0 versus the total energy in units of μ with the subtraction at $\omega = \mu$.

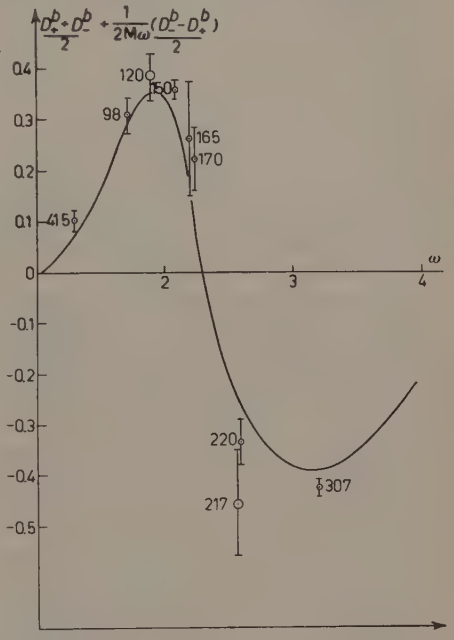

Fig. 2. – $(D^b_- + D^b_+)/2r_0 + (1/2M\omega)\cdot(D^b_- - D^b_+)/2r_0$ versus the total energy.

Comparing with the P.S. results, an improvement in the region below and around the resonance may be noticed. No appreciable improvement is noticed in the region above the resonance. But one must take into account that all points have been obtained with an analysis of the angular distribution made only with S and P waves. As Hamilton and Chiu [11] have shown, the points around 300 MeV, analyzed with S, P and D waves, have large errors in their real part of the scattering amplitude, and in addition give values rather different from those obtained analyzing only with S and P waves. Therefore these points cannot discriminate between the various D^- curves above the resonance. Thus, the only points which remain in disagreement with a value of f^2 around 0.08 are the two Ashkin points [12] at 150 and 170 MeV.

In spite of the fact that Hamilton has suggested that one can resolve this disagreement by modifying the total cross section used in the optical theorem at these two energies, we have mantained the P.S. values of D^- because at the moment no new experimental data are available in this region. Fig. 2, in which is plotted

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[8] G. SALZMAN: private communication to prof. PUPPI.
[9] H. L. ANDERSON and N. METROPOLIS: *Proc. of the Sixth Annual Rochester Conference* (1956).
[10] R. COOL, O. PICCIONI and D. CLARK: *Phys. Rev.*, **103**, 1082 (1956).
[11] J. HAMILTON and H. Y. CHIU: *Phys. Rev. Lett.*, **1**, 4 (1958).
[12] J. ASHKIN, J. P. BLASER, F. FEINER and M. O. STERN: *Phys. Rev.*, **101**, 1149 (1956).
[13] J. HAMILTON: *Phys. Rev.*, **110**, 1134 (1958).

the relationship

$$\frac{D_+^b + D_-^b}{2r_0} + \frac{1}{2M\omega}\cdot\frac{D_-^b - D_+^b}{2r_0}\,.$$

in which the f^2 term disappears, shows in a clear way the discrepancy between the 150 and 170 MeV points and the total ensemble of experimental data.

In this analysis, as well as in that of P.S., the subtraction is made at the total energy $\omega = \mu$. This way of handling the dispersion relations may introduce at least two sources of trouble:

1) As CHEW *et al.* [14] have pointed out, the P.S. analysis was not completely from free the hypothesis of charge independence, because they used Orear's scattering lengths, which were obtained from elastic scattering experiments as well as from charge exchange experiments.

2) Because of the mass differences between charged and neutral pions, and between protons and neutrons, the total σ cross section is different from zero at $\omega = \mu$, so that the integrals appearing in the dispersion relations are meaningless, except if one makes particular assumptions about the behaviour of the imaginary part of the scattering amplitude in the region between μ_0 and μ, assumptions which we prefer to avoid.

For these reasons, it is more convenient to make the subtraction at a total energy different from μ, at which the real parts of the scattering amplitudes are

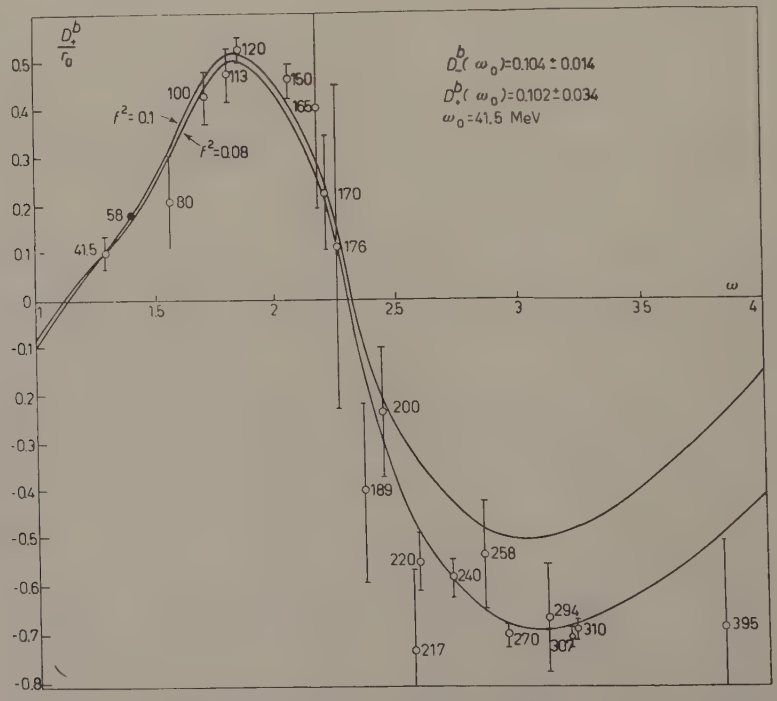

Fig. 3. – D_+^b/r_0 versus the total energy with the subtraction at $\omega = 41.5$ MeV.

[14] G. F. CHEW and H. P. NOYES: *Phys. Rev.*, **109**, 566 (1958).

experimentally known. Such an analysis has already been made by CHIU [15], who has chosen the kinetic energy 150 MeV as the subtraction energy. However, this particular energy is not very useful for the study of the region around the resonance, since all the curves are forced to pass through the subtraction point, independently of the value of f^2.

So we have chosen as subtraction point the Rochester 41.5 MeV point [6], and we have calculated the two relations:

$$\frac{D^b_{\pm}}{r_0} = \frac{\eta_b}{\eta}(\eta^2 - \eta_0^2)\left[\pm \frac{2f^2}{\omega \pm 1/2\,M} \cdot \frac{(1 - 1/4\,M^2)}{(\omega_0^2 - 1/4\,M^2)} + \frac{1}{4\pi^2 r_0^2}\int_1^{\infty} \frac{\eta'\,\mathrm{d}\omega'}{\omega'^2 - \omega_0^2}\left(\frac{\sigma^+}{\omega' \mp \omega} + \frac{\sigma^-}{\omega' \pm \omega}\right) + \right.$$

$$\left. + \frac{1}{2}\,\frac{\eta_0}{\eta_{0b}}\left(\left(1 + \frac{\omega}{\omega_0}\right)\frac{D^b_{\pm}(\eta_0)}{r_0} + \left(1 - \frac{\omega}{\omega_0}\right)\frac{D^b_{\mp}(\eta_0)}{r_0}\right)\right].$$

The results are shown in Figs. 3 and 4.

For the D^+ relations the situation is the same as P.S. found, although the curves are less sensitive to the value of f^2. The D^- relation, in comparison with the P.S.

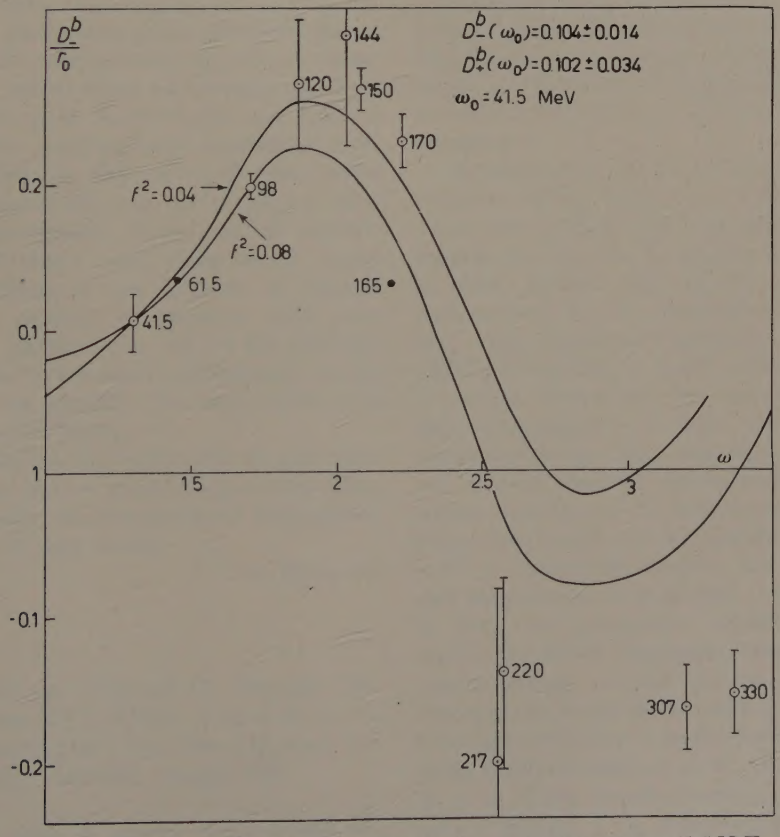

Fig. 4. – D^b_-/r_0 versus the total energy with the subtraction at $\omega = 41.5$ MeV.

[15] H. Y. CHIU: *Phys. Rev.*, **110**, 1140 (1958).

curves, shows some improvements, there being essential agreement with our result obtained subtracting at $\omega = \mu$. Unfortunately, in this way, the continuous curves have errors which arise from the real parts at fixed energy which are much larger than those errors arising from the scattering lengths in the $\omega = \mu$ subtraction case. Thus one cannot obtain definite conclusions from dispersion relations used in this manner, until more precise experiments are finished in the low energy region.

In conclusion, from the present values of the experimental results one can say that the general behaviour of the data agrees with the prediction of dispersion relations and is compatible with a coupling constant f^2 around 0.08. There is a remaining discrepancy in the region between 150 and 200 MeV, which cannot be solved with further manipulations of the dispersion relations, but only with new experimental results. Thus, until these new experiments are finished, one cannot worry about the origin of this discrepancy or look for a violation of some fundamental principle.

* * *

We wish to thank prof. G. Puppi and Drs. A. Minguzzi and A. Stanghellini for stimulating discussions.

LIBRI RICEVUTI E RECENSIONI

R. J. BLIN-STOYLE – *Theories of Nuclear Moments*. Pagg. 89, fig. 12, Oxford University Press, 1957

Come dice il titolo stesso, e come è espresso nella prefazione, scopo di questo volumetto è l'inquadramento teorico dei risultati sperimentali concernenti i momenti nucleari in base ai vari modelli del nucleo.

In questo senso esso potrebbe essere un utile complemento ai testi di argomento affine, come ad esempio quelli di Ramsey e di Kopfermann, nei quali i momenti nucleari sono considerati prevalentemente dal punto di vista della loro interazione con campi interni (atomici, molecolari, cristallini), od esterni.

Purtroppo però l'argomento viene condensato in un numero di pagine esiguo rispetto all'ampiezza delle questioni trattate, per cui c'è da dubitare che un lettore non specializzato possa trarre un effettivo vantaggio dalla sola lettura del libro.

L'utilità del volumetto si può riconoscere se lo si considera piuttosto come una guida alla consultazione della bibliografia in esso citata.

G. BOLOGNA

Reports on Progress in Physics, Volume XXI (1958), a cura di A. C. STICKLAND; Ed. The Physical Society, London, pagg. 383.

La mole di questo ventunesimo volume della raccolta di lavori monografici pubblicata dalla Physical Society di Londra è alquanto minore di quella del volume precedente, segnando così una battuta di arresto nella serie rapidamente crescente di quelli pubblicati negli ultimi anni. Ciò non significa tuttavia che l'interesse della pubblicazione possa considerarsi minore, perchè essa conserva l'alto livello nel contenuto e l'accuratezza nella presentazione che sempre l'hanno distinta.

La scelta degli argomenti, che vanno dalla biofisica alla fisica nucleare, dalle tecniche di laboratorio agli sviluppi della teoria, è fatta come sempre dando la preferenza a problemi moderni e di largo interesse.

È notevole in questo volume il posto occupato dalla Fisica dei Solidi alla quale sono connesse ben tre delle nove monografie; in una di queste dedicata ai calori specifici dei metalli a bassa temperatura D. H. PARKINSON, dopo aver separatamente esaminato il contributo reticolare e quello elettronico, si occupa brevemente dei casi specifici dei superconduttori e delle terre rare, concludendo con una esposizione delle più recenti tecniche sperimentali. Uno studio sugli spettri di assorbimento e di fluorescenza degli ioni nei cristalli dovuto a W. A. RUNCIMAN riunisce poi i risultati più recenti ottenuti per i vari tipi di ioni con particolare riguardo allo studio dei livelli energetici. Sempre in questo gruppo si può poi considerare l'accurata rassegna critica sull'analisi delle strutture cristalline a mezzo della diffrazione degli elettroni di J. M. COWLEY e A. L. G. REES, in cui si cerca di coordinare i metodi e la nomenclatura illustrando in particolare l'applicazione dei metodi di Fourier.

Passando agli altri argomenti ricor-

diamo che M. BRINK in un articolo di carattere essenzialmente teorico si occupa dei recenti progressi relativi alla interpretazione della struttura del nucleo, con particolare riguardo agli aspetti « collettivi », mentre R. HUBY tratta della eccitazione elettromagnetica dei nuclei a mezzo di proiettili nucleari o di elettroni, con una esposizione equamente bilanciata della parte teorica e di quella sperimentale.

Un lavoro sugli atomi mesonici (D. WEST) ne passa in rassegna le caratteristiche principali insieme alla informazione che se ne può trarre sulle proprietà dei mesoni e dei nuclei, tra loro interagenti; anche in questo lavoro è fatto il debito luogo sia alla discussione dei dati sperimentali che alla teoria.

J. M. VALENTINE ed S. C. CURRAN si occupano poi dell'energia mediamente spesa per ogni coppia di ioni prodotti in un gas da vari tipi di radiazione descrivendo gli orientamenti più recenti di questi studi e fornendo interessanti tabelle di risultati.

La produzione e la misura di altissime temperature hanno fatto importanti progressi connessi anche con i modernissimi studi sulla fusione controllata; W. LOCHTE-HOLTGREVEN ci offre qui un quadro, notevole per la sua completezza, dei metodi che possono essere impiegati e dei risultati ottenuti; la produzione di alte temperature con mezzi chimici è esposta in appendice da R. SCHALL e F. WECKEN.

Finalmente la fisica del sistema nervoso, un tipico esempio di collaborazione fra le scienze fisiche e quelle biologiche, è trattata da P. FATT, uno specialista di biofisica dello University College di Londra.

F. A. LEVI

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